# Limity funkcí I

1. Dokažte z definice, že
   a) $\lim\limits_{x\to 1}\left(\dfrac{x}{2}\right)^3=\dfrac{1}{8}$ b) $\lim\limits_{x\to 1^+}[x]=1$ c) $\lim\limits_{x\to 1^-}[x]=0$

   Spočtěte

2. (a) $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1}$ (b) $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1}$

3. $\lim\limits_{x\to 2}\left(\dfrac{1}{x^2-2x}-\dfrac{x}{x^2-4}\right)$

4. $\lim\limits_{x\to 0}\dfrac{(1+x)(1+2x)\dots(1+nx)-1}{x}$, $n\in\mathbb{N}$

5. $\lim\limits_{x\to 1}\dfrac{x^{100}-2x+1}{x^{50}-2x+1}$

6. $\lim\limits_{x\to 0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}$, $m,n\in\mathbb{N}$

7. $\lim\limits_{x\to 1}\dfrac{x^{n+1}-(n+1)x+n}{(x-1)^2}$, $n\in\mathbb{N}$

8. $\lim\limits_{x\to 1}\dfrac{x+x^2+\dots x^n-n}{x-1}$, $n\in\mathbb{N}$

9. $\lim\limits_{x\to 1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$, $m,n\in\mathbb{N}$

10. $\lim\limits_{x\to 0}\dfrac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}}$

11. $\lim\limits_{x\to 0+}\dfrac{(\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1})}{x}$

12. $\lim\limits_{x\to 0^+}\left(\sqrt{\dfrac{1}{x}+\sqrt{\dfrac{1}{x}+\sqrt{\dfrac{1}{x}}}}-\sqrt{\dfrac{1}{x}-\sqrt{\dfrac{1}{x}+\sqrt{\dfrac{1}{x}}}}\right)$

13. (a) $\lim_{x\to 16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$ (b) $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$

14. $\lim_{x\to 0} \dfrac{\sqrt{1-2x-x^2}-(1-x)}{x}$

15. $\lim_{x\to 0} \dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}}$

16. $\lim_{x\to 0} \dfrac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}$, $m, n \in \mathbb{N}$

17. $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}$

18. $\lim_{x\to a^+} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$, $a \in \mathbb{R}_0^+$

19. $\lim_{x\to 0} \dfrac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1}{x}$, $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}$

# Limity funkcí

**Definice:** Řekneme, že $\lim_{x\to c} f(x) = A \overset{\text{def.}}{\iff} \forall \varepsilon>0\ \exists \delta>0 : \forall x\in\mathbb{R}:$

$$x\in P(c,\delta) \Longrightarrow f(x)\in U(A,\varepsilon)$$

Zde $P(c,\delta)$ je prstencové okolí bodu $c$ o poloměru $\delta$, tedy $(c-\delta,c)\cup(c,c+\delta)$

$U(A,\varepsilon)$ je úplné okolí bodu $A$ o poloměru $\varepsilon$, tedy $(A-\varepsilon, A+\varepsilon)$

Tedy jinými slovy totéž: $\forall\varepsilon>0\ \exists\delta>0\ \forall x\in\mathbb{R}: \ 0<|x-c|<\delta \Rightarrow |f(x)-A|<\varepsilon$

Jak vidno, nepředpokládám nic o existenci nebo hodnotě $f(c)$ !!

Limity zleva a zprava analogicky pro levé a pravé prstencové okolí $c$, tj. $P_-(c,\delta)$, $P_+(c,\delta)$

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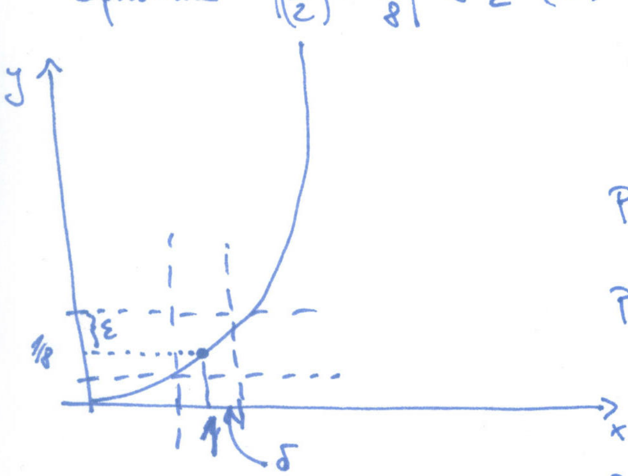
(1) Dokažte z definice

a) $\lim_{x\to 1} \left(\frac{x}{2}\right)^3 = \frac{1}{8}$. Musíme ukázat: $\forall\varepsilon>0\ \exists\delta>0\ \forall x\in\mathbb{R}: 0<|x-1|<\delta \Rightarrow \left|\left(\frac{x}{2}\right)^3-\frac{1}{8}\right|<\varepsilon$

Nechť tedy soupeř zvolí $\varepsilon>0$ libovolně malé. My musíme najít $\delta>0$ tak malé (v závislosti na $\varepsilon$ samozřejmě), aby pro $|x-1|<\delta$ platilo $\left|\left(\frac{x}{2}\right)^3-\frac{1}{8}\right|<\varepsilon$.

Upravíme $\left|\left(\frac{x}{2}\right)^3-\frac{1}{8}\right|<\varepsilon \iff |x^3-1|<8\varepsilon$




Nejvyšší hodnoty dosáhne funkce $|x^3-1|$ na $P(1,\delta)$ v bodech $1-\delta$ nebo $1+\delta$.

Pro $1-\delta$: $|(1-\delta)^3-1| = |-3\delta+3\delta^2-\delta^3|$
$= 3\delta-3\delta^2+\delta^3$

Pro $1+\delta$: $|(1+\delta)^3-1| = |3\delta+3\delta^2+\delta^3|$
$= 3\delta+3\delta^2+\delta^3$

$1+\delta$ mi dá vyšší hodnotu.

Hledám $\delta$ tak, že $3\delta+3\delta^2+\delta^3<8\varepsilon$. Pokud $\varepsilon\geq 1$, stačí zvolit $\delta=\frac{1}{10}$.
To není zajímavé (ale je potřeba to napsat). Nechť tedy $\varepsilon<1$.
Budu hledat $\delta<1$ a proto $\delta^2<\delta$ a $\delta^3<\delta$.

Takto $3\delta+3\delta^2+\delta^3<3\delta+3\delta+\delta=7\delta$. Pokud najdu takové $\delta<1$, že $7\delta<8\varepsilon$, jsem hotov. Tj. $\delta<\frac{8}{7}\varepsilon$ a můžu zvolit $\delta=\varepsilon$

Pro $\delta=\varepsilon$ mám $3\varepsilon+3\varepsilon^2+\varepsilon^3<7\varepsilon<8\varepsilon$ a tedy $\forall x\in\mathbb{R}: 0<|x-1|<\varepsilon \Rightarrow$
$\Rightarrow \left|\left(\frac{x}{2}\right)^3-\frac{1}{8}\right|<\varepsilon$
(pro $\varepsilon<1$)

b) $\lim\limits_{x\to 1+} [x] = 1$. Definice celé části: $[x]=n$ pro $n \le x < n+1$ $(n\in\mathbb{Z})$ (2)

Definice limity zprava: $\forall \varepsilon>0\ \exists\delta>0\ \forall x\in\mathbb{R}: 0<x-1<\delta \Rightarrow |[x]-1|<\varepsilon$

Očividně platí: $[x]=1$ pro $x\in[1,2)$. Tedy mohu zvolit $\delta$ libovolné menší než 1

např. $\delta=1/2$ bez ohledu na ~~[illegible]~~ $\varepsilon>0$, které volí soupeř.

Pro $0<x-1<1/2$ platí $[x]-1=0$ a $0<\varepsilon$ pro lib. $\varepsilon\,(>0)$.

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c) $\lim\limits_{x\to 1-} [x] = 0$.

Limita zleva: $\forall \varepsilon>0\ \exists\delta>0\ \forall x\in\mathbb{R}: 0<1-x<\delta \Rightarrow |[x]|<\varepsilon$

Opět, na levém okolí jedničky je fce $[x]$ konstantní, platí $[x]=0$ pro $x\in[0,1)$

Mohu zase zvolit $\delta=1/2$ a bude platit $0<1-x<1/2 \Rightarrow [x]=0$

a tedy $|[x]|<\varepsilon$

pro lib. $\varepsilon>0$.

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2) a) $\lim\limits_{x\to 0} \dfrac{x^2-1}{2x^2-x-1}$. Zkusím dosadit $x=0$: $\dfrac{0^2-1}{2\cdot 0^2-0-1} = \dfrac{-1}{-1} = 1$.

Tedy $f(0)=1$ pro $f(x)=\dfrac{x^2-1}{2x^2-x-1}$. Také platí, že $f$ je spojitá na okolí nuly (Tato $f$ je nespojitá jen v bodech, kde jmenovatel $=0$). Proto $\lim\limits_{x\to 0} f(x) = f(0) = 1$.

b) $\lim\limits_{x\to 1} \dfrac{x^2-1}{2x^2-x-1}$. Zkouška dosazení $x=1$: $\dfrac{1^2-1}{2\cdot 1^2-1-1} = \dfrac{0}{0}$ :o(

Ale mám podíl polynomů, jejichž kořen je $x=1$. Platí proto

$$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \underbrace{\frac{(x-1)(x+1)}{(x-1)(2x+1)}}_{f(x)} = \lim_{x\to 1} \underbrace{\frac{x+1}{2x+1}}_{g(x)} = \underline{\underline{\frac{2}{3}}}$$

Tyto 2 fce jsou totožné pro $x\neq 1$,

proto $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} g(x)$

3) $$\lim_{x\to 2}\left(\frac{1}{x^2-2x}-\frac{x}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{1}{x(x-2)}-\frac{x}{(x-2)(x+2)}\right)=$$

$$=\lim_{x\to 2}\frac{x+2-x^2}{x(x-2)(x+2)}=\lim_{x\to 2}\frac{(x-2)(-x-1)}{x(x-2)(x+2)}=\lim_{x\to 2}\frac{-x-1}{x(x+2)}=\underline{\underline{\frac{-3}{8}}}$$

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4) $$\lim_{x\to 0}\frac{(1+x)(1+2x)\cdot\ldots\cdot(1+nx)-1}{x}=\lim_{x\to 0}\frac{1+x\cdot(1+2+\ldots+n)+x^2\cdot z(x)-1}{x}=$$

$$=\lim_{x\to 0}\frac{n(n+1)}{2}+x\cdot z(x)=\underline{\underline{\frac{n(n+1)}{2}}}$$

$z(x)$ je polynom $(n-2)$. stupně

$$\lim_{x\to 0}x\cdot z(x)=0\cdot z(0)=0$$

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5) $$\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1}$$

Zde by se hodilo vytknout $(x-1)$, to ale nevypadá lehce. Pomůžeme si trikem

$y:=x-1$, tedy $x=y+1$. Potom $\lim_{x\to 1}f(x)=\lim_{y\to 0}f(y+1)$.

Tedy $$\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1}=\lim_{y\to 0}\frac{(y+1)^{100}-2(y+1)+1}{(y+1)^{50}-2(y+1)+1}=\lim_{y\to 0}\frac{(y+1)^{100}-2y-1}{(y+1)^{50}-2y-1}=$$

$$=\lim_{y\to 0}\frac{1+100\cdot y+y^2z_1(y)-2y-1}{1+50\cdot y+y^2z_2(y)-2y-1}=\lim_{y\to 0}\frac{98y+y^2z_1(y)}{48y+y^2z_2(y)}=\lim_{y\to 0}\frac{98+y\,z_1(y)}{48+y\,z_2(y)}=\underline{\underline{\frac{49}{24}}}$$

Zde opět $z_1(y)$, $z_2(y)$ jsou polynomy

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6) $$\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2}=\lim_{x\to 0}\frac{1+mnx+\frac{n(n-1)}{2}m^2x^2+x^3z_1(x)-\left(1+mnx+\frac{m(m-1)}{2}n^2x^2+x^3z_2(x)\right)}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{m^2n(n-1)}{2}-\frac{m(m-1)n^2}{2}+x\cdot(z_1(x)-z_2(x))\right)=\frac{mn}{2}(n-m)$$

. Výsledek platí i pro $m=n$ nebo pro $m=1,2$ či $n=1,2$.

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7) $$\lim_{x\to 1}\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}=\lim_{y\to 0}\frac{(y+1)^{n+1}-(n+1)(y+1)+n}{y^2}=\lim_{y\to 0}\frac{(y+1)^{n+1}-(n+1)y-1}{y^2}=$$

$$=\lim_{y\to 0}\frac{1+(n+1)y+\frac{(n+1)n}{2}y^2+y^3z(y)-1-(n+1)y}{y^2}=\lim_{y\to 0}\frac{(n+1)n}{2}+yz(y)=\underline{\underline{\frac{(n+1)n}{2}}}$$

(Platí i pro $n=1$)

8) $$\lim_{x\to 1} \frac{x+x^2+\dots+x^n-n}{x-1} = \lim_{x\to 1}\left(\frac{x-1}{x-1}+\frac{x^2-1}{x-1}+\dots+\frac{x^n-1}{x-1}\right)=$$

$$=\lim_{x\to 1}\left(1+(x+1)+(x^2+x+1)+\dots+(x^{n-1}+x^{n-2}+\dots+x+1)\right) = 1+2+3+\dots+n = \underline{\underline{\frac{n(n+1)}{2}}}$$

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9) $$\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\lim_{x\to 1}\frac{m(1-x^n)-n(1-x^m)}{(1-x^m)(1-x^n)}=$$

$$=\lim_{y\to 0}\frac{m\left(1-(y+1)^n\right)-n\left(1-(y+1)^m\right)}{\left(1-(y+1)^m\right)\left(1-(y+1)^n\right)}=\lim_{y\to 0}\frac{m\left(-ny-\frac{n(n-1)}{2}y^2-y^3z_2(y)\right)-n\left(-my-\frac{m(m-1)}{2}y^2-y^3z_1(y)\right)}{\left(my+\frac{m(m-1)}{2}y^2+y^3z_1(y)\right)\left(ny+\frac{n(n-1)}{2}y^2+y^3z_2(y)\right)}$$

$$=\lim_{y\to 0}\frac{\left(\frac{nm(m-1)}{2}-\frac{mn(n-1)}{2}\right)y^2+y^3\left(nz_1(y)-mz_2(y)\right)}{mny^2+y^3z_3(y)}=\lim_{y\to 0}\frac{\frac{(m-n)}{2}\cdot mn+y\cdot z_4(y)}{mn+y\cdot z_3(y)}=$$

$$=\underline{\underline{\frac{m-n}{2}}}$$

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10) $$\lim_{x\to 0}\frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}}=\lim_{x\to 0}\frac{2+x^2}{\sqrt{3-6x^2+5x^4}}=\underline{\underline{\frac{2}{\sqrt{3}}=\frac{2}{3}\sqrt{3}}}$$

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11) $$\lim_{x\to 0_+}\frac{\sqrt{1/x^2+1}-\sqrt{1/x^2-1}}{x}=\lim_{x\to 0_+}\frac{\frac{\sqrt{1+x^2}}{x}-\frac{\sqrt{1-x^2}}{x}}{x}=\lim_{x\to 0_+}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}=$$

$$=\lim_{x\to 0_+}\frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}=\lim_{x\to 0_+}\frac{2x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}=\frac{2}{2}=\underline{\underline{1}}$$

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12) $$\lim_{x\to 0_+}\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{1/x}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{1/x}}}=\lim_{x\to 0_+}\frac{2\sqrt{\frac{1}{x}+\sqrt{1/x}}}{\sqrt{\frac{1}{x}+\sqrt{1/x+\sqrt{1/x}}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{1/x}}}}=$$

$$=\lim_{x\to 0_+}\frac{2\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}}{\sqrt{\frac{1}{x}+\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}}+\sqrt{\frac{1}{x}-\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}}}=\lim_{x\to 0_+}\frac{2\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}}{\frac{\sqrt{1+\sqrt{x}\sqrt{1+\sqrt{x}}}}{\sqrt{x}}+\frac{\sqrt{1-\sqrt{x}\sqrt{1+\sqrt{x}}}}{\sqrt{x}}}=$$

$$=\lim_{x\to 0_+}\frac{2\sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x}\sqrt{1+\sqrt{x}}}+\sqrt{1-\sqrt{x}\sqrt{1+\sqrt{x}}}}=\frac{2}{2}=\underline{\underline{1}}$$

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13) a) $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4} = \lim\limits_{x\to 16} \dfrac{(\sqrt[4]{x}-2)(\sqrt[4]{x}+2)}{(\sqrt{x}-4)\cancel{(\sqrt{x}+4)}(\sqrt[4]{x}+2)} = \lim\limits_{x\to 16} \dfrac{1}{\sqrt[4]{x}+2} = \underline{\underline{\dfrac{1}{4}}}$

b) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x} = \lim\limits_{x\to 0} \dfrac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x+1}+1} = \underline{\underline{\dfrac{1}{2}}}$

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14) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-2x-x^2}-(1-x)}{x} = \lim\limits_{x\to 0} \dfrac{\left(\sqrt{1-2x-x^2}-\sqrt{1-2x+x^2}\right)\left(\sqrt{1-2x-x^2}+\sqrt{1-2x+x^2}\right)}{x\cdot\left(\sqrt{1-2x-x^2}+\sqrt{1-2x+x^2}\right)} =$

$= \lim\limits_{x\to 0} \dfrac{-2x^2}{x\cdot\left(\sqrt{1-2x-x^2}+\sqrt{1-2x+x^2}\right)} = \lim\limits_{x\to 0} \dfrac{-2x}{\sqrt{1-2x-x^2}+\sqrt{1-2x+x^2}} = \underline{\underline{0}}$

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15) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}} = \lim\limits_{x\to 0} \dfrac{2x}{\left(x+2x^{4/3}\right)\left(\sqrt[3]{(27+x)^2}+\sqrt[3]{(27-x)(27+x)}+\sqrt[3]{(27-x)^2}\right)} =$

$= \lim\limits_{x\to 0} \dfrac{2}{\left(1+2\sqrt[3]{x}\right)\left(\sqrt[3]{(27+x)^2}+\sqrt[3]{(27-x)(27+x)}+\sqrt[3]{(27-x)^2}\right)} = \dfrac{2}{3\cdot 9} = \underline{\underline{\dfrac{2}{27}}}$

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16) $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+x}-1}{x} + \dfrac{1-\sqrt[n]{1+x}}{x} = \lim\limits_{x\to 0} \dfrac{1+x-1}{x\cdot\left((1+x)^{\frac{m-1}{m}}+\ldots+1\right)} +$

$+ \lim\limits_{x\to 0} \dfrac{1-(1+x)}{x\cdot\left(1+\ldots+(1+x)^{\frac{n-1}{n}}\right)} = \underline{\underline{\dfrac{1}{m}-\dfrac{1}{n}}}$

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17) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}} = \lim\limits_{x\to 0} \dfrac{\sqrt[6]{(1+x)^3}-\sqrt[6]{(1-x)^2}}{\sqrt[6]{(1+x)^2}-\sqrt[6]{(1-x)^3}} = \lim\limits_{x\to 0} \dfrac{\left[(1+x)^3-(1-x)^2\right]\left[\left(\sqrt[6]{(1+x)^2}\right)^5+\left(\sqrt[6]{(1+x)^2}\right)^4\cdot\sqrt[6]{(1-x)^3}+\ldots\right]}{\left[(1+x)^2-(1-x)^3\right]\left[\left(\sqrt[6]{(1+x)^3}\right)^5+\left(\sqrt[6]{(1+x)^3}\right)^4\cdot\sqrt[6]{(1-x)^2}+\ldots\right]}$

$= \lim\limits_{x\to 0} \dfrac{1+3x+3x^2+x^3-1+2x-x^2}{1+2x+x^2-1+3x-3x^2+x^3}\cdot\dfrac{[\ldots]}{[\ldots]} = \lim\limits_{x\to 0} \dfrac{5+2x+x^3}{5-2x+x^3}\cdot\dfrac{[\ldots]}{[\ldots]} = \dfrac{5\cdot 6}{5\cdot 6} = \underline{\underline{1}}$

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18) $\lim\limits_{x\to a_+} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} = \lim\limits_{x\to a_+} \dfrac{x-a}{\sqrt{x^2-a^2}\cdot(\sqrt{x}+\sqrt{a})} + \dfrac{1}{\sqrt{x+a}} = \lim\limits_{x\to a_+} \dfrac{\sqrt{x-a}}{\sqrt{x+a}\cdot(\sqrt{x}+\sqrt{a})} + \dfrac{1}{\sqrt{x+a}}$

$= \dfrac{1}{\sqrt{2a}} = \underline{\underline{\dfrac{\sqrt{2a}}{2a}}}$

19) $$\lim_{x\to 0} \frac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx} - 1}{x} = \lim_{x\to 0} \frac{\sqrt[nm]{(1+ax)^n(1+bx)^m} - 1}{x} =$$

ozn. $y=(1+ax)^n(1+bx)^m$

$$= \lim_{x\to 0} \frac{(1+ax)^n(1+bx)^m - 1}{x} \cdot \frac{1}{1+y^{1/nm}+y^{2/nm}+\dots y^{\frac{nm-1}{nm}}}$$

$$= \lim_{x\to 0} \frac{(1+anx+x^2z_1(x))(1+bmx+x^2z_2(x)) - 1}{x} \cdot \frac{1}{1+y^{1/nm}+\dots+y^{\frac{nm-1}{nm}}} =$$

$$= \lim_{x\to 0} \frac{x\cdot(an+bm)+x^2z_3(x)}{x} \cdot \frac{1}{1+\dots+y^{\frac{nm-1}{nm}}} = \frac{an+bm}{nm}$$