NEWTONŮV A RIEMANNŮV INTEGRÁL

(1)
$$\int_0^{\ln 2}\sqrt{e^x-1}\,dx = \left[\begin{array}{l} e^x-1=t \quad x=\ln(t+1) \\ dx=\frac{1}{t+1}\,dt \\ e^0-1=0 \\ e^{\ln 2}-1=1\end{array}\right] = \int_0^1 \frac{\sqrt{t}}{t+1}\,dt \overset{y=\sqrt{t},\ t=y^2,\ dt=2y\,dy}{=} \int_0^1 \frac{2y^2}{y^2+1}\,dy$$
$$= \int_0^1 2-\frac{2}{y^2+1}\,dy = 2\left[y-\operatorname{arctg} y\right]_0^1 = 2(1-0-\operatorname{arctg}1+\operatorname{arctg}0) = 2\left(1-\frac{\pi}{4}\right)$$

(2)
$$\int_0^1 1\cdot\arccos x\,dx \overset{PP}{=} \left[x\arccos x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = 0-\left[\sqrt{1-x^2}\right]_0^1 = 1$$
$u'=1 \quad v=\arccos x$
$u=x \quad v'=-\frac{1}{\sqrt{1-x^2}}$

(3)
$$\int_0^\infty x^{2k-1}e^{-\frac{x^2}{2}}\,dx,\quad k\in\mathbb{N}$$
$$\| \qquad I_k$$
$$\to \lim_{x\to\infty}\left(-e^{-\frac{x^2}{2}}\right)-\left(-e^{-\frac{0^2}{2}}\right) \qquad 2k-3=2(k-1)-1$$
$$I_1=\int_0^\infty xe^{-\frac{x^2}{2}}\,dx = -\left[e^{-\frac{x^2}{2}}\right]_0^\infty = 1$$
$$k>1\quad I_k=\int_0^\infty x^{2k-1}e^{-\frac{x^2}{2}}\,dx = \int_0^\infty x^{2k-2}\,xe^{-\frac{x^2}{2}}\,dx \overset{PP}{=} -\left[x^{2k-2}e^{-\frac{x^2}{2}}\right]_0^\infty + (2k-2)\int_0^\infty x^{2k-3}e^{-\frac{x^2}{2}}\,dx$$
$$= (2k-2)\,I_{k-1} = 2(k-1)\,I_{k-1}$$
$u=x^{2k-2}$
$u'=(2k-2)x^{2k-3}$
$v'=xe^{-\frac{x^2}{2}}$
$v=-e^{-\frac{x^2}{2}}$

$$\Rightarrow I_k = 2^{k-1}(k-1)!$$

(4)
$$\int_0^{4\pi}\frac{1}{1+\sin^2 x}\,dx = 8\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 x}\,dx = 8\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{1+\sin^2 x}\,\frac{dx}{\cos^2 x}$$
$$= 8\int_0^{\frac{\pi}{2}}\frac{1}{\frac{\cos^2 x+\sin^2 x+\sin^2 x}{\cos^2 x}}\,\frac{dx}{\cos^2 x} = 8\int_0^{\frac{\pi}{2}}\frac{1}{1+2\operatorname{tg}^2 x}\,\frac{dx}{\cos^2 x} \overset{y=\operatorname{tg}x,\ dy=\frac{dx}{\cos^2 x}}{=} 8\int_0^\infty\frac{1}{1+2y^2}\,dy$$
$$= 4\sqrt{2}\left[\operatorname{arctg}\sqrt{2}y\right]_0^\infty = 2\sqrt{2}\pi$$

(5)
$$\int_0^{2\pi}\frac{1}{\sin^4 x+\cos^4 x}\,dx = 4\int_0^{\frac{\pi}{2}}\frac{1}{\sin^4 x+\cos^4 x}\,dx \overset{y=\operatorname{tg}x,\ dx=\frac{dy}{y^2+1}}{=} 4\int_0^\infty \frac{1}{\left(\frac{y^2}{1+y^2}\right)^2+\left(\frac{1}{1+y^2}\right)^2}\cdot\frac{1}{1+y^2}\,dy$$
$$= 4\int_0^\infty\frac{1+y^2}{y^4+1}\,dy = 4\int_0^\infty\frac{y^2+1}{(y^2+1-\sqrt{2}y)(y^2+1+\sqrt{2}y)}\,dy$$
$$= 2\int_0^\infty\frac{1}{y^2+1-\sqrt{2}y}+\frac{1}{y^2+1+\sqrt{2}y}\,dy = 4\int_0^\infty\frac{1}{2\left(y-\frac{\sqrt{2}}{2}\right)^2+1}+\frac{1}{2\left(y+\frac{\sqrt{2}}{2}\right)^2+1}\,dy$$
$$= \frac{4}{\sqrt{2}}\left[\operatorname{arctg}\left(\sqrt{2}\left(y-\frac{\sqrt{2}}{2}\right)\right)+\operatorname{arctg}\left(\sqrt{2}\left(y+\frac{\sqrt{2}}{2}\right)\right)\right]_0^\infty = 2\sqrt{2}\pi$$

⑥ $\int_2^\infty \frac{1}{x^2}dx = -\left[\frac{1}{x}\right]_2^\infty = \frac{1}{2}$ → $-\left(\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{2}\right)\right)$

⑦ $\int_0^\infty e^{-3x}dx = -\frac{1}{3}\left[e^{-3x}\right]_0^\infty = \frac{1}{3}$

⑧ $\int_0^1 x\ln x\,dx \overset{pp}{=} \left[\frac{x^2}{2}\ln x\right]_0^1 - \int_0^1 \frac{x}{2} = 0 - \left[\frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$

$u'=x \quad v=\ln x$

$u=\frac{x^2}{2} \quad v'=\frac{1}{x}$

$\frac{1^2}{2}\ln 1 - \lim_{x\to 0+}\frac{x^2}{2}\ln x$

⑨ $\int_0^\infty e^{-ax}\cos bx$ $\quad \boxed{a>0}$ $\quad b=0 \Rightarrow \int_0^\infty e^{-ax}dx = \left[-\frac{e^{-ax}}{a}\right]_0^\infty = \frac{1}{a}$

$b\neq 0 \quad I = \int_0^\infty e^{-ax}\cos bx = \left[-\frac{e^{-ax}}{a}\cos bx\right]_0^\infty - \int_0^\infty \frac{e^{-ax}}{a}b\sin bx$

$v'=e^{-ax} \quad u=\cos bx$

$v=-\frac{e^{-ax}}{a} \quad u'=-b\sin bx$

$v'=\frac{e^{-ax}}{a} \quad u=b\sin bx$

$u'=b^2\cos bx \quad v=-\frac{e^{-ax}}{a^2}$

$= \frac{1}{a} - \left[-\frac{e^{-ax}}{a^2}b\sin bx\right]_0^\infty - \int_0^\infty \frac{e^{-ax}}{a^2}b^2\cos bx$

$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{1}{a}$

$= \frac{1}{a} - \frac{b^2}{a^2}I \Rightarrow I = \frac{a}{a^2+b^2}$

⑩ $\int_0^{\frac{\pi}{2}} \operatorname{tg} x\,dx = \lim_{a\to\frac{\pi}{2}} \int_0^a \operatorname{tg} x\,dx = \lim_{a\to\frac{\pi}{2}} -\left[\ln(\cos x)\right]_0^a = \lim_{a\to\frac{\pi}{2}}(-\ln\cos a) = \infty$

⑪ $F(\alpha) = \int_0^\pi \ln(1-2\alpha\cos x+\alpha^2)\,dx \quad |\alpha|\neq 1$

$F'(\alpha) = \int_0^\pi \frac{2\alpha - 2\cos x}{1-2\alpha\cos x+\alpha^2}dx$ → Cmil'm

$F(0)=0 \quad F(\alpha) = \int_0^\alpha F'(\alpha) = \int_0^\alpha\left(\int_0^\pi \frac{2\alpha-2\cos x}{1-2\alpha\cos x+\alpha^2}dx\right)d\alpha$

z definice Horní a dolní součet na $(0,\pi)$ $\quad x_0=0,\ x_1=\frac{\pi}{n},\ \dots\dots\ x_n=\pi$

$I_n = \frac{1}{n}\sum_{k=1}^{n}\ln(1-2\alpha\cos x_k+\alpha^2)$ $\quad ; \frac{1}{n}\sum_{k=1}^{n}\ln(1-2\alpha\cos x_{k-1}+\alpha^2) = y_n$

Horní dolní součty

Spočteme pouze jeden součet:

$\frac{1}{n}\sum_{k=0}^{n-1}\ln\left(1-2\alpha\cos\frac{\pi k}{n}+\alpha^2\right) = \frac{1}{n}\sum_{k=0}^{n-1}\ln\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi k}{n}\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi}{n}k\right)\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi k}{n}\right)\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi(k-2n)}{n}+i\sin\frac{\pi(k-2n)}{n}\right)\right)$

$= \frac{1}{n}\ln(\ \ \ \ \cdot\cdot -$

$= \frac{1}{n}\ln(\ \cdot\cdot - \quad \prod_{k=n+1}^{2n}$

$\underbrace{\left(\alpha-\cos\left(\frac{-k\pi}{n}\right)+i\sin\left(-\frac{\pi k}{n}\right)\right)}$

$\left(\alpha-\cos\left(\frac{\pi k}{n}\right)-i\sin\left(\frac{\pi k}{n}\right)\right)$

$$= \frac{1}{n} \ln \left(\prod_{\substack{k=0 \\ k\neq n}}^{2n} \left(\alpha - \cos\frac{k\pi}{n} - i \sin\frac{k\pi}{n} \right) \right)$$

$$= \frac{1}{n} \ln \left(\left[\prod_{k=0}^{2n-1} \left(\alpha - \cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n} \right) \right] \frac{\alpha - \cos\frac{2n\pi}{n} - i\sin\frac{2n\pi}{n}}{\alpha - \cos\frac{n\pi}{n} - i\sin\frac{n\pi}{n}} \right)$$

$$= \frac{1}{n} \ln \left(\frac{\alpha-1}{\alpha+1} \prod_{k=0}^{2n-1} (\alpha - z_k) \right) \qquad \text{kde } z_k = \cos\frac{k\pi}{n} + i \sin\frac{k\pi}{n}$$

ALE $(z_k)^{2n} = \cos 2k\pi + i \sin 2k\pi = 1$

a tedy $\prod_{k=0}^{2n-1} (\alpha - z_k) = \alpha^{2n} - 1$

$$\Rightarrow \; = \frac{1}{n} \ln \left(\frac{\alpha-1}{\alpha+1} (\alpha^{2n} - 1) \right) \qquad (\text{máme } |\alpha| \neq 1 \;!)$$

a tedy $\int_0^\pi \ln(1 - 2\alpha\cos x + \alpha^2) = \lim_{n\to\infty} \frac{1}{n} \ln \left(\frac{\alpha-1}{\alpha+1} (\alpha^{2n}-1) \right)$

POKUD $|\alpha| < 1$ pak $= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \ln \left(\frac{1-\alpha}{\alpha+1} (1 - \alpha^{2n}) \right)$

$$= 0 \cdot \ln\left(\frac{1-\alpha}{1+\alpha} \right) = 0$$

POKUD $|\alpha| > 1$ pak $= \lim_{n\to\infty} \left(\frac{1}{n} \ln \frac{\alpha-1}{\alpha-1} + \frac{1}{n} \ln (\alpha^{2n} - 1) \right)$

$$= \lim_{n\to\infty} \frac{1}{n} \ln \left(\alpha^{2n} \frac{\alpha^{2n}-1}{\alpha^{2n}} \right) = \lim_{n\to\infty} \left(\frac{1}{n} \ln \alpha^{2n} + \frac{1}{n} \ln \left(\frac{\alpha^{2n}-1}{\alpha^{2n}} \right) \right)$$

$$= \underline{\underline{\ln \alpha^2}}$$

KONVERGENCE INTEGRÁLU – T] "$\int_a^b f(x)\,dx < \infty$" ?

(12) $\int_0^\infty x^p\,dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^{\frac{1}{\varepsilon}} x^p\,dx = \lim_{\varepsilon\to 0+} \left[\frac{x^{p+1}}{p+1} \right]_\varepsilon^{\frac{1}{\varepsilon}} \underset{p\neq 1}{} = \lim_{\varepsilon\to 0+} \frac{1}{p+1} \left(\varepsilon^{-(p+1)} - \varepsilon^{p+1} \right) = \infty$

$$= \lim_{\varepsilon\to 0+} \left[\ln x \right]_\varepsilon^{1/\varepsilon} \underset{p=1}{} = \lim_{\varepsilon\to 0+} \ln \frac{1}{\varepsilon^2} = \infty$$

<u>NEKONVERGUJE!</u>

(13) $\int_1^\infty x^p\,dx = \lim_{\varepsilon\to 0+} \left[\frac{x^{p+1}}{p+1} \right]_1^{\frac{1}{\varepsilon}} \underset{p\neq 1}{} = \begin{cases} -\frac{1}{p+1} & \text{PRO } p < -1 \\ \infty & \text{PRO } p > -1 \end{cases}$

$$= \lim_{\varepsilon\to 0+} [\ln x]_1^{\frac{1}{\varepsilon}} = \infty$$

KONVERGUJE $\iff p < -1$! Důležitý!

$\int_2^\infty \frac{1}{x \ln^p x}\,dx = \lim_{\varepsilon\to 0+} \frac{1}{p-1} \left[\frac{1}{\ln^{p-1} x} \right]_2^{\frac{1}{\varepsilon}} \underset{p\neq 1}{} \begin{cases} \text{KONVERGUJE} & p > 1 \\ \text{DIVERGUJE} & p < 1 \end{cases}$

$\lim_{\varepsilon\to 0} \left[\frac{1}{\ln(\ln x)} \right]_2^{1/\varepsilon} \underset{p=1}{}$ – DIVERGUJE

$\int_{10}^\infty \frac{1}{x \ln x (\ln(\ln x))^p}$

(14) $\int_0^{10} x^p = \lim_{\varepsilon \to 0+} \left[\frac{x^{p+1}}{p+1} \right]_\varepsilon^{10} \begin{cases} \frac{10^{p+1}}{p+1} & p > -1 \\ \infty & p < -1 \end{cases}$

$p = -1 \Rightarrow [\ln x]_\varepsilon^{10} \Rightarrow \infty$

KONVERGUJE $\Leftrightarrow$ $p > -1$

[INTEGRÁL KONVERGUJE "v ∞" pokud $f(x) \sim x^p$ pro $p < -1$

"v 0" —||— $p > -1$]

(15) $\int_0^\infty \frac{x^{3/2}}{1+x^2}\,dx$ jediný problémový bod je "∞"

PROTOŽE $\frac{x^{3/2}}{1+x^2} \sim x^{-1/2}$ $x \to \infty$

a $\int_1^\infty x^{-1/2}\,dx = \infty$ [$\int_0^\infty$ —||— $\geq \int_1^\infty \frac{x^{3/2}}{1+x^2}\,dx \geq \int_1^\infty \frac{x^{3/2}}{2x^2} = \frac{1}{2}\int_1^\infty x^{-\frac{1}{2}}\,dx = \infty$]

pak $\int_0^\infty \frac{x^{-1/2}}{1+x^2}\,dx = \infty$

(16) $\int_0^1 \frac{1}{\sqrt{x(1-x^2)}}\,dx = \int_0^1 \underbrace{\frac{1}{\sqrt{1+x}} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}}_{f(x)}\,dx$ PROBLÉMOVÉ BODY $x=0$, $x=1$

v okolí "0" $f(x) \sim \frac{1}{\sqrt{x}}$ a $\int_0^1 \frac{1}{\sqrt{x}} < \infty$

v okolí "1" $f(x) \sim \frac{1}{\sqrt{1-x}}$ a $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx < \infty$

$\Rightarrow \int_0^1 f(x) < \infty$ [$\int_0^1$ —||— $= \int_0^{1/2} + \int_{1/2}^1 \leq \int_0^{1/2} 1 \cdot \frac{1}{\sqrt{1/2}} \frac{1}{\sqrt{x}} + \int_{1/2}^1 1 \cdot \frac{1}{\sqrt{1/2}} \frac{1}{\sqrt{1-x}} < \infty$]

(17) $\int_0^2 \frac{1}{\ln x}\,dx$ PROBLÉMOVÝ BOD "$x=1$"

$\ln x \sim (x-1)$ v "$x=1$" jinak je $\frac{1}{\ln x}$ spojitá

a $\int_1^2 \frac{1}{x-1}\,dx = \infty$ $\Rightarrow$ INTEGRÁL NEKONVERGUJE!

(18) $\int_0^{1/2} \frac{\ln \sin x}{x^p}\,dx$ PROBLÉMOVÝ BOD $x=0$

$\ln \sin x \sim \ln x$

POKUD $p \geq 1$ pak $\frac{|\ln x|}{x^p} \geq \frac{1}{x}$ v okolí "0" a integrál diverguje

pokud $p < 1$ pak $\frac{|\ln x|}{x^p} \lesssim \frac{1}{x^{1-\varepsilon}}$ pro nějaké $\varepsilon > 0$

$\Rightarrow$ KONVERGUJE

(19) $\int_0^\infty \frac{\operatorname{arctg} x}{x^{3/2}} = \int_1^\infty \frac{\operatorname{arctg} x}{x^{3/2}} + \int_0^1 \frac{\operatorname{arctg} x}{x^{3/2}} \leq \int_1^\infty \frac{\frac{\pi}{2}}{x^{3/2}} + \int_0^1 \frac{x}{x^{3/2}} < \infty$

$\int \frac{1}{x^{3/2}}$, $\int \frac{1}{x^{1/2}}$