Sada příkladů 11/11

Fourierova transformace temperovaných distribucí

Dokažte tabulku Fourierových transformací.

Fourierova transformace distribucí

(1): $f \in L^1(\mathbb{R}^N)$
$\mathcal{F}(f) = \int_{\mathbb{R}^N} f(x) \exp\{-2\pi\mathrm{i}(x,\xi)\}\,\mathrm{d}x$

(2): $\mathbb{A} \in \mathbb{R}^{N\times N}$, poz. definitní, symetrická
$\mathcal{F}(\exp\{-(\mathbb{A}x,x)\}) = \frac{(\sqrt{\pi})^N}{\sqrt{|\det\mathbb{A}|}} \exp\{-\pi^2(\mathbb{A}^{-1}\xi,\xi)\}$

(3): $\delta \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}(\delta) = 1$

(4): $T_1 \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}(T_1) = \delta$

(5): $T_{x^n} \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}(T_{x^n}) = \frac{1}{(-2\pi\mathrm{i})^n} D^n\delta$

(6): $D^n\delta \in \mathcal{S}'(\mathbb{R})$, $n \in \mathbb{N}$
$\mathcal{F}(D^n\delta) = (2\pi\mathrm{i})^n\xi^n$

(7): $b \in \mathbb{C}$
$\mathcal{F}\left(T_{\exp(2\pi\mathrm{i}bx)}\right) = \delta_b$

(8): $b \in \mathbb{C}$
$\mathcal{F}\left(T_{\sin(2\pi bx)}\right) = \frac{1}{2\mathrm{i}}(\delta_b - \delta_{-b})$

(9): $b \in \mathbb{C}$
$\mathcal{F}\left(T_{\cos(2\pi bx)}\right) = \frac{1}{2}(\delta_b + \delta_{-b})$

(10): $b \in \mathbb{C}$
$\mathcal{F}\left(T_{\sinh(2\pi bx)}\right) = \frac{1}{2}(\delta_{-\mathrm{i}b} - \delta_{\mathrm{i}b})$

(11): $b \in \mathbb{C}$
$\mathcal{F}\left(T_{\cosh(2\pi bx)}\right) = \frac{1}{2}(\delta_{-\mathrm{i}b} + \delta_{\mathrm{i}b})$

(12): $H_{x_+^\lambda} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$
$\mathcal{F}\left(\frac{H_{x_+^\lambda}}{\Gamma(\lambda+1)}\right) = \frac{\mathrm{e}^{-\mathrm{i}(\lambda+1)\frac{\pi}{2}}}{(2\pi)^{\lambda+1}} H_{(\xi-\mathrm{i}0)^{-\lambda-1}}$

(13): $x_+^n \in \mathcal{S}'(\mathbb{R})$, $n \in \mathbb{N}$
$\mathcal{F}\left(H_{x_+^n}\right) = (2\pi\mathrm{i})^{-n-1} n! H_{\xi^{-n-1}} + \frac{1}{2}(2\pi\mathrm{i})^{-n}(-1)^{-n}D^n\delta$

(14): $T_H \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}(T_H) = \mathcal{F}\left(H_{x_+^0}\right) = \frac{1}{2\pi\mathrm{i}} T_{\mathrm{v.p.}\,\xi^{-1}} + \frac{1}{2}\delta$

(15): $H_{x_-^\lambda} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$
$\mathcal{F}\left(\frac{H_{x_-^\lambda}}{\Gamma(\lambda+1)}\right) = \mathrm{e}^{\mathrm{i}(\lambda+1)\frac{\pi}{2}}(2\pi)^{-\lambda-1} H_{(\xi+\mathrm{i}0)^{-\lambda-1}}$

(16): $H_{|x|^\lambda} = H_{x_+^\lambda} + H_{x_-^\lambda} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$, $\lambda \neq -n$, $n \in \mathbb{N}_0$
$\mathcal{F}\left(H_{|x|^\lambda}\right) = -2\Gamma(\lambda+1)(2\pi)^{-\lambda-1}\sin\left(\frac{\pi}{2}\lambda\right) H_{|\xi|^{-\lambda-1}}$

(17): $H_{|x|^\lambda \operatorname{sign} x} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$; $\lambda \neq -n$, $n \in \mathbb{N}_0$
$\mathcal{F}\left(H_{|x|^\lambda \operatorname{sign} x}\right) = -2\mathrm{i}\frac{\Gamma(\lambda+1)}{(2\pi)^{\lambda+1}}\cos\left(\frac{\pi}{2}\lambda\right) H_{|\xi|^{-\lambda-1}\operatorname{sign}\xi}$

(18): $H_{x^{-m}} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$; $\quad \mathcal{F}(H_{x^{-m}})$
$= \begin{cases} (-1)^{\frac{m+1}{2}}\mathrm{i}\pi\frac{(2\pi)^{m-1}}{(m-1)!} H_{|\xi|^{m-1}\operatorname{sign}\xi} & m \text{ liché} \\ (-1)^{\frac{m}{2}}\frac{\pi(2\pi)^{m-1}}{(m-1)!} H_{|\xi|^{m-1}} & m \text{ sudé} \end{cases}$

(19): $T_{\mathrm{v.p.}\,x^{-1}} \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}\left(T_{\mathrm{v.p.}\,x^{-1}}\right) = -\mathrm{i}\pi T_{\operatorname{sign}\xi}$

(20): $T_{\operatorname{sign} x} \in \mathcal{S}'(\mathbb{R})$
$\mathcal{F}(T_{\operatorname{sign} x}) = \frac{1}{\mathrm{i}\pi} T_{\mathrm{v.p.}\,x^{-1}}$

(21): $H_{x^{-2}} \in \mathcal{S}'(\mathbb{R})$,
$\mathcal{F}(H_{x^{-2}}) = -T_{|\xi|} 2\pi^2$

(22): $H_{(x+\mathrm{i}0)^\lambda} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$
$\mathcal{F}\left(H_{(x+\mathrm{i}0)^\lambda}\right) = \frac{H_{\xi_+^{-\lambda-1}}}{\Gamma(-\lambda)} \exp\{\mathrm{i}\lambda\frac{\pi}{2}\}(2\pi)^{-\lambda}$

(23): $H_{(x-\mathrm{i}0)^\lambda} \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{C}$
$\mathcal{F}\left(H_{(x-\mathrm{i}0)^\lambda}\right) = \frac{H_{\xi_-^{-\lambda-1}}}{\Gamma(-\lambda)} \exp\{-\mathrm{i}\lambda\frac{\pi}{2}\}(2\pi)^{-\lambda}$

(24): $r = |x|, x \in \mathbb{R}^N$, $\lambda \in \mathbb{C}$, $\quad \rho = |\xi|, \xi \in \mathbb{R}^N$
$\mathcal{F}\left(\frac{H_{r^\lambda}}{\Gamma(\frac{\lambda+N}{2})}\right) = \frac{H_{\rho^{-\lambda-N}}}{\Gamma(-\lambda/2)\pi^{\lambda+N/2}}$

TEMPEROVANÉ DISTRIBUCE

Už víme: $S(\mathbb{R}^N) = \{ f \in C^\infty(\mathbb{R}^N) : \|f\|_{\alpha,\beta} = \|x^\alpha D^\beta f\|_{L^\infty} < \infty \ \forall \alpha,\beta \in \mathbb{N}_0^N \}$

$$f_n \xrightarrow{S} f \iff \|f_n - f\|_{\alpha\beta} \to 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^N$$

$S'(\mathbb{R}^N)$ ... prostor temper. distribucí ... množina spojitých lin. fcionálů nad $S(\mathbb{R}^N)$, kde spojitost je

$$\varphi_k \xrightarrow{S} 0 \implies \langle T, \varphi_k \rangle \to 0$$

a $T_k \xrightarrow{*} T$ v S' $\iff \langle T_k, \varphi \rangle \to \langle T, \varphi \rangle \quad \forall \varphi \in S(\mathbb{R}^N)$

$D \subset S \implies S' \subset D'$, tj. temperované distribuce jsou speciální případ distribucí

(Např. pro $f = e^{2x^2}$ je $T_f \in D'(\mathbb{R})$ ale $T_f \notin S'(\mathbb{R})$)

Platí: $f \in L^1_{loc}(\mathbb{R}^N)$ a $|f(x)| \le C(1+|x|^2)^m \implies T_f \in S'(\mathbb{R}^N)$

$f \in L^p(\mathbb{R}^N)$ pro $p \in [1,\infty] \implies T_f \in S'(\mathbb{R}^N)$

Pomalu rostoucí fce: $\Theta_M := \{ f \in C^\infty(\mathbb{R}^N) : \forall \alpha \in \mathbb{N}_0^N \ \exists m_\alpha \in \mathbb{N}_0 \ \exists c_\alpha > 0 : |D^\alpha f(x)| \le c_\alpha (1+|x|^2)^{m_\alpha} \}$

Je-li $a \in \Theta_M$, pak $\varphi \mapsto a\varphi$ spojité zobrazují $S(\mathbb{R}^N)$ do $S(\mathbb{R}^N)$.

Definujeme: $T \in S'(\mathbb{R}^N)$, $\alpha \in \mathbb{N}_0^N$: $D^\alpha T \in S'(\mathbb{R}^N)$: $\langle D^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in S(\mathbb{R}^N)$

$a \in \Theta_M$: $aT \in S'(\mathbb{R}^N)$: $\langle aT, \varphi \rangle := \langle T, a\varphi \rangle \quad \forall \varphi \in S(\mathbb{R}^N)$

$A \in \mathbb{R}^{N\times N}$ regulární, $b \in \mathbb{R}^N$: $\langle T(Ay+b), \varphi(y) \rangle := \langle T(x), \frac{1}{|\det A|} \cdot \varphi(A^{-1}(x-b)) \rangle \ \forall \varphi$

Fourierova transformace: $T \in S'(\mathbb{R}^N)$. Definujeme $\mathcal{F}(T) \in S'(\mathbb{R}^N)$ a $\mathcal{F}^{-1}(T) \in S'(\mathbb{R}^N)$ jako

$$\langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle$$
$$\langle \mathcal{F}^{-1}(T), \varphi \rangle := \langle T, \mathcal{F}^{-1}(\varphi) \rangle \quad \forall \varphi \in S(\mathbb{R}^N).$$

Platí $\mathcal{F}^{-1}(\mathcal{F}(T)) = T \ \forall T \in S'$ a $\mathcal{F}(\mathcal{F}^{-1}(T)) = T$

$\mathcal{F}$ a $\mathcal{F}^{-1}$ jsou spojitá zobrazení z $S'(\mathbb{R}^N)$ do $S'(\mathbb{R}^N)$

Platí: $D^\alpha(\mathcal{F}(T)) = \mathcal{F}((-2\pi i x)^\alpha T)$ a $\mathcal{F}(D^\alpha T) = (2\pi i \xi)^\alpha \mathcal{F}(T)$

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1) $f \in L^1(\mathbb{R}^N)$ a uvažujme $T_f \in S'(\mathbb{R}^N)$

$$\langle \mathcal{F}(T_f), \varphi \rangle = \langle T_f, \mathcal{F}(\varphi) \rangle = \int_{\mathbb{R}^N} f(x) \mathcal{F}(\varphi)(x)\, dx = \int_{\mathbb{R}^N} f(x) \left(\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} \varphi(\xi)\, d\xi \right) dx \overset{\text{Fubini}}{=}$$

$$= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x) \varphi(\xi) e^{-2\pi i x \cdot \xi}\, dx\, d\xi \overset{\text{Fubini}}{=} \int_{\mathbb{R}^N} \varphi(\xi) \left(\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} f(x)\, dx \right) d\xi$$

$$= \int_{\mathbb{R}^N} \varphi(\xi) \mathcal{F}(f)(\xi)\, d\xi = \langle T_{\mathcal{F}(f)}, \varphi \rangle \quad \forall \varphi \in S(\mathbb{R}^N)$$

Dokázali jsme $\mathcal{F}(T_f) = T_{\mathcal{F}(f)}$

2) $A \in \mathbb{R}^{N\times N}$ poz. def., symetrická $\Rightarrow$ ex. $B \in \mathbb{R}^{N\times N}$ poz. def., sym. t.ž. $A = BB$

Pak $e^{-Ax\cdot x} = e^{-Bx\cdot Bx} =: f(x)$. Zajímá nás $\langle \mathcal{F}T_f, \varphi\rangle = \langle T_f, \mathcal{F}\varphi\rangle =$

$$= \int_{\mathbb{R}^N} e^{-Bx\cdot Bx}\left(\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\varphi(\xi)\,d\xi\right)dx = \int_{\mathbb{R}^N}\varphi(\xi)\int_{\mathbb{R}^N} e^{-[Bx\cdot Bx + 2\pi i Bx\cdot B^{-1}\xi]}\,dx = \left| \begin{array}{l} y = Bx \\ \Rightarrow dx = \frac{dy}{\det B}\end{array}\right.$$

$$= \int_{\mathbb{R}^N}\varphi(\xi)\int_{\mathbb{R}^N} e^{-[y\cdot y + 2\pi i y\cdot B^{-1}\xi - \pi^2 B^{-1}\xi\cdot B^{-1}\xi]}\cdot e^{-\pi^2 B^{-1}\xi\cdot B^{-1}\xi}\cdot\frac{dy}{\det B} \quad \left| \begin{array}{l} \text{“}z = y + \pi i B^{-1}\xi\text{”} \\ dz = dy\end{array}\right.$$

$$= \int_{\mathbb{R}^N}\varphi(\xi)\,\frac{e^{-\pi^2 B^{-1}B^{-1}\xi\cdot\xi}}{\det B}\cdot\int_{\mathbb{R}^N} e^{-z\cdot z}\,dz = \int_{\mathbb{R}^N}\frac{\pi^{N/2}}{\sqrt{\det A}}\,e^{-\pi^2 A^{-1}\xi\cdot\xi}\varphi(\xi)\,d\xi = \langle T_g,\varphi\rangle \text{ pro}$$

$$g(\xi) = \frac{\pi^{N/2}}{\sqrt{\det A}}\,e^{-\pi^2 A^{-1}\xi\cdot\xi}$$

Viz také příklad 3, z cvičení 10, kde je zdůvodněno, že posunutí v imaginárním směru nemění hodnotu integrálu e^{-x^2}.

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3) $\langle \mathcal{F}\delta_0,\varphi\rangle = \langle\delta_0,\mathcal{F}\varphi\rangle = \left(\int_{\mathbb{R}} e^{-2\pi i x\xi}\varphi(x)\,dx\right)\Big|_{\xi=0} = \int_{\mathbb{R}}\varphi(x)\,dx = \langle T_1,\varphi\rangle$

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4) $\langle\mathcal{F}T_1,\varphi\rangle = \langle T_1,\mathcal{F}\varphi\rangle = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2\pi i x\xi}\varphi(x)\,dx\right)d\xi$ Nelze použít Fubiniho!

Místo toho: $\langle\mathcal{F}^{-1}\delta_0,\varphi\rangle = \langle\delta_0,\mathcal{F}^{-1}\varphi\rangle = \left(\int_{\mathbb{R}} e^{2\pi i x\xi}\varphi(x)\,dx\right)\Big|_{\xi=0} = \int_{\mathbb{R}}\varphi(x)\,dx = \langle T_1,\varphi\rangle$

a použijeme $\mathcal{F}(\mathcal{F}^{-1}\delta_0) = \delta_0 \Rightarrow \mathcal{F}(T_1) = \delta_0$

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5) Pro $f(x)=x^n$ je $\mathcal{F}T_f = \mathcal{F}(x^n) = \mathcal{F}(x^n T_1) = \frac{1}{(-2\pi i)^n}\mathcal{F}\left((-2\pi i x)^n T_1\right) = \frac{1}{(-2\pi i)^n}D^n\mathcal{F}(T_1)$

$$= \frac{1}{(-2\pi i)^n}D^n\delta_0$$

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6) Pro $T = D^n\delta_0$ je $\mathcal{F}T = \mathcal{F}(D^n\delta_0) = (2\pi i\xi)^n\mathcal{F}(\delta_0) = (2\pi i\xi)^n T_1 = (2\pi i\xi)^n$

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7) $f(x) = e^{2\pi i b x}$. Standardní postup opět nefunguje: $\langle T_f,\mathcal{F}\varphi\rangle = \int_{\mathbb{R}} e^{2\pi i b x}\left(\int_{\mathbb{R}} e^{-2\pi i x\xi}\varphi(\xi)\,d\xi\right)dx \overset{?}{=}$

$\overset{?}{=}\int_{\mathbb{R}}\varphi(\xi)\underbrace{\left(\int_{\mathbb{R}} e^{2\pi i x(b-\xi)}\,dx\right)}_{\text{toto nemá integrál!}}d\xi$. Musíme jinak, použijeme spojitost Fourierovy transformace v S'.

Víme: $e^{2\pi i b x} = \sum_{n=0}^{\infty} \frac{(2\pi i b x)^n}{n!}$. Označíme-li $T_N = \sum_{0}^{N} \frac{(2\pi i b x)^n}{n!}$, pak $T_N \xrightarrow{*} T$ v S'

a proto $\mathcal{F}(T_N)\xrightarrow{*}\mathcal{F}(T)$.

$\langle\mathcal{F}(T_N),\varphi\rangle$ = dle 5) = $\sum_{n=0}^{N}\frac{(2\pi i b)^n}{n!}\langle\mathcal{F}x^n,\varphi\rangle = \sum_{0}^{N}\frac{(2\pi i b)^n}{n!}\cdot\frac{1}{(-2\pi i)^n}\langle D^n\delta_0,\varphi\rangle = \sum_{0}^{N}\frac{(-b)^n}{n!}(-1)^n\langle\delta_0, D^n\varphi\rangle$

$= \sum_{0}^{N}\frac{b^n}{n!}\varphi^{(n)}(0)$.... To je Taylorův rozvoj $\varphi(b) = \varphi(0)+b\varphi'(0)+\frac{b^2}{2}\varphi''(0)+\dots$

Tj. $\lim_{N\to\infty}\langle\mathcal{F}(T_N),\varphi\rangle = \varphi(b) = \langle\delta_b,\varphi\rangle$ To ale funguje jen pro analytické fce a $\varphi\in S$ nemusí být analytická.

Nejjednodušeji: $\langle \mathcal{F}^{-1}\delta_b, \varphi\rangle = \langle \delta_b, \mathcal{F}^{-1}\varphi\rangle = \left(\int_{\mathbb{R}} e^{2\pi i x\xi}\varphi(x)\,dx\right)\Big|_{\xi=b} = \int_{\mathbb{R}} e^{2\pi i x b}\varphi(x)\,dx =$

$= \langle T_{e^{2\pi i x b}}, \varphi\rangle$

Protože $\mathcal{F}(\mathcal{F}^{-1}\delta_b) = \delta_b$ je $\mathcal{F}(T_{e^{2\pi i x b}}) = \delta_b$.

8) $\sin(2\pi b x) = \frac{1}{2i}\left(e^{2\pi i b x} - e^{-2\pi i b x}\right)$, $\mathcal{F}$ je lineární $\Rightarrow$ použitím příkladu 7 dostanu

$\mathcal{F}(T_{\sin(2\pi b x)}) = \frac{1}{2i}\left(\mathcal{F}(T_{e^{2\pi i b x}}) - \mathcal{F}(T_{e^{-2\pi i b x}})\right) = \frac{1}{2i}(\delta_b - \delta_{-b})$

9) $\cos(2\pi b x) = \frac{1}{2}\left(e^{2\pi i b x} + e^{-2\pi i b x}\right)$ a stejně jako 8) využijeme 7): $\mathcal{F}(T_{\cos(2\pi b x)}) = \frac{1}{2}(\delta_b + \delta_{-b})$

10) $\sinh(2\pi b x) = \frac{1}{2}\left(e^{2\pi b x} - e^{-2\pi b x}\right) = \frac{1}{2}\left(e^{-2\pi i (ib) x} - e^{2\pi i (ib) x}\right)$ a stejně jako výše využijeme 7)

$\mathcal{F}(T_{\sinh(2\pi b x)}) = \frac{1}{2}(\delta_{-ib} - \delta_{ib})$

11) $\cosh(2\pi b x) = \frac{1}{2}\left(e^{2\pi b x} + e^{-2\pi b x}\right) = \frac{1}{2}\left(e^{-2\pi i (ib) x} + e^{2\pi i (ib) x}\right) \Rightarrow \mathcal{F}(T_{\cosh(2\pi b x)}) = \frac{1}{2}(\delta_{-ib} + \delta_{ib})$

12) Toto je detailně vysvětleno ve skriptech na začátku kapitoly 24.4.

13) To je speciální případ 12) pro $\lambda = n \in \mathbb{N}$

Zde $\Gamma(n+1) = n!$ a dále $\exp(-i(n+1)\frac{\pi}{2}) = -i\cdot\exp(-in\frac{\pi}{2}) = -i\cdot(-i)^n = (-i)^{n+1} = \left(\frac{1}{i}\right)^{n+1}$
$= i^{-n-1}$

Zbývá určit $H_{(\xi-i0)^{-n-1}}$, to ale víme z př. 11 z minulé série:

$$H_{(\xi-i0)^{-n-1}} = H_{\xi^{-n-1}} + \frac{i\pi(-1)^n}{n!}D^n\delta_0$$

Dohromady tak $\mathcal{F}(H_{x_+^n}) = n!(2\pi i)^{-n-1}H_{\xi^{-n-1}} + \underbrace{n!(2\pi i)^{-n-1}\cdot\frac{i\pi(-1)^n}{n!}}_{=\frac{1}{2}(-2\pi i)^{-n}}D^n\delta_0$

14) To je speciální případ 13), $H(x) = x_+^0$, tj. $n = 0$

$$\mathcal{F}(T_H) = \frac{1}{2\pi i}H_{\xi^{-1}} + \frac{1}{2}\delta_0 = \frac{1}{2\pi i}T_{\mathrm{p.v.}\,1/\xi} + \frac{1}{2}\delta_0$$

15) Stejně jako 12) s přirozenými modifikacemi, ta největší spočívá v tom, že se volí $\sigma < 0$, aby $\mathcal{F}(x_-^\lambda e^{-2\pi\sigma x})(\xi) = \int_{-\infty}^{0} x^\lambda e^{-2\pi(\sigma+i\xi)x}\,dx = \int_0^\infty y^\lambda e^{-2\pi(|\sigma|-i\xi)y}\,dy$

Stejným výpočtem pak $\mathcal{F}(x_-^\lambda e^{-2\pi\sigma x})(\xi) = \left(2\pi(-\sigma-i\xi)^{-\lambda-1}\right)\Gamma(\lambda+1)$

$$= \frac{\Gamma(\lambda+1)}{(2\pi)^{\lambda+1}}\cdot(-i)^{-\lambda-1}\cdot(\xi-i\sigma)^{-\lambda-1}$$

$$= \frac{\Gamma(\lambda+1)}{(2\pi)^{\lambda+1}}\cdot e^{(-i\frac{\pi}{2})(-\lambda-1)}\cdot\underbrace{(\xi+i|\sigma|)^{-\lambda-1}}_{\to(\xi+i0)^{-\lambda-1}}$$

16) Sečtením 12, a 15, dostaneme

$$\Gamma(\lambda+1)\cdot(2\pi)^{-\lambda-1}\cdot\underbrace{\left[e^{-i(\lambda+1)\pi/2} H_{(\xi-i0)^{-\lambda-1}} + e^{i(\lambda+1)\pi/2} H_{(\xi+i0)^{-\lambda-1}}\right]}_{(*)} = \mathcal{F}(H_{|x|^\lambda})$$

$(*)$ = (z definice $(\xi\pm i0)^\lambda$) $= e^{-i(\lambda+1)\pi/2}\cdot\left[H_{\xi_+^{-\lambda-1}} + e^{i(\lambda+1)\pi} H_{\xi_-^{-\lambda-1}}\right] + e^{i(\lambda+1)\pi/2}\cdot\left[H_{\xi_+^{-\lambda-1}} + e^{-i(\lambda+1)\pi}H_{\xi_-^{-\lambda-1}}\right] =$

$$= \left(H_{\xi_+^{-\lambda-1}} + H_{\xi_-^{-\lambda-1}}\right)\left(e^{-i(\lambda+1)\pi/2} + e^{i(\lambda+1)\pi/2}\right)$$

$$= H_{|\xi|^{-\lambda-1}}\cdot 2\cos\left(\lambda\tfrac{\pi}{2}+\tfrac{\pi}{2}\right) = H_{|\xi|^{-\lambda-1}}\cdot(-2)\cdot\sin\left(\lambda\tfrac{\pi}{2}\right)$$

17) Odečtením 12, a 15, dostaneme

$$\mathcal{F}(H_{|x|^\lambda \operatorname{sgn} x}) = \Gamma(\lambda+1)(2\pi)^{-\lambda-1}\cdot \underbrace{\left[e^{-i(\lambda+1)\pi/2}H_{(\xi-i0)^{-\lambda-1}} - e^{i(\lambda+1)\pi/2} H_{(\xi+i0)^{-\lambda-1}}\right]}_{(*)}$$

$$(*) = e^{-i(\lambda+1)\pi/2}\left[H_{\xi_+^{-\lambda-1}} + e^{i(\lambda+1)\pi}H_{\xi_-^{-\lambda-1}}\right] - e^{i(\lambda+1)\pi/2}\left[H_{\xi_+^{-\lambda-1}} + e^{-i(\lambda+1)\pi}H_{\xi_-^{-\lambda-1}}\right] =$$

$$= H_{\xi_+^{-\lambda-1}}\cdot\left(e^{-i(\lambda+1)\pi/2} - e^{i(\lambda+1)\pi/2}\right) - H_{\xi_-^{-\lambda-1}}\left(e^{-i(\lambda+1)\pi/2} - e^{i(\lambda+1)\pi/2}\right) =$$

$$= H_{|\xi|^{-\lambda-1}\operatorname{sgn}\xi}\cdot(-2i)\sin\left(\lambda\tfrac{\pi}{2}+\tfrac{\pi}{2}\right) = H_{|\xi|^{-\lambda-1}\operatorname{sgn}\xi}\cdot(-2i)\cos\left(\lambda\tfrac{\pi}{2}\right)$$

18) m sudé: $m=2n$. Všimne, že $H_{x^{-2n}} = \lim_{\lambda\to-2n} H_{|x|^\lambda}$ a dále, že F.T. je spojitá, proto

$$\mathcal{F}(H_{x^{-2n}}) = \lim_{\lambda\to-2n}\mathcal{F}(H_{|x|^\lambda}) \overset{16)}{=} \lim_{\lambda\to-2n}\Gamma(\lambda+1)\cdot(2\pi)^{-\lambda-1}\cdot(-2)\cdot H_{|\xi|^{-\lambda-1}}\cdot\sin\left(\lambda\tfrac{\pi}{2}\right)$$

$$= -2\cdot(2\pi)^{2n-1}\cdot(-2n)\cdot\left(\lim_{\lambda\to-2n}\Gamma(\lambda)\cdot\sin\lambda\tfrac{\pi}{2}\right) H_{|\xi|^{2n-1}}$$

$$= 2\cdot m\cdot(2\pi)^{m-1} H_{|\xi|^{m-1}}\cdot\left[\lim_{\lambda\to-2n}\Gamma(\lambda)\cdot(-1)^n\cdot\sin\left(\left(\tfrac{\lambda}{2}+n\right)\pi\right)\right]$$

$$= (-1)^{m/2}\cdot 2m(2\pi)^{m-1}H_{|\xi|^{m-1}}\cdot\lim_{\lambda\to-2n}\Gamma(\lambda)\cdot\frac{\sin\left(\left(\frac{\lambda}{2}+n\right)\pi\right)}{\left(\frac{\lambda}{2}+n\right)\pi}\cdot\frac{1}{2}(\lambda+2n)\pi$$

$$= (-1)^{m/2}\cdot m\pi(2\pi)^{m-1}H_{|\xi|^{m-1}}\cdot\underbrace{\lim_{\lambda\to-2n}\Gamma(\lambda)\cdot(\lambda+2n)} = \operatorname{Res}_{-2n}\Gamma = \frac{(-1)^{2n}}{(2n)!} = \frac{1}{m!}$$

$$= (-1)^{m/2}\pi(2\pi)^{m-1}\cdot\frac{1}{(m-1)!}H_{|\xi|^{m-1}}$$

m liché: $m=2n+1$. Analogicky s použitím příkladu 17, Opět použijeme $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$.

Dále $\cos\left(\lambda\frac{\pi}{2}\right) = \sin\left(\frac{(\lambda+1)\pi}{2}\right) = (-1)^n\sin\left(\left(\frac{\lambda+1}{2}+n\right)\pi\right) \approx (-1)^n\cdot\frac{1}{2}\cdot\pi\cdot(\lambda+1+2n)$

Dále $\lim_{\lambda\to-2n-1}\Gamma(\lambda)(\lambda+1+2n) = \operatorname{Res}_{-2n-1}\Gamma = \frac{(-1)^{2n+1}}{(2n+1)!} = \frac{-1}{m!}$

$$\boxed{(-1)^{\frac{m-1}{2}}\cdot(-1)\cdot\frac{\pi}{2}\cdot\frac{1}{m!}} = (-1)^{\frac{m+1}{2}}$$

a zbytek je zřejmý

19) Speciální případ 18) pro $m=1$. Lze ale řešit dohromady s 20) takto:

20) Uvažujme posloupnost distribucí $T_{\operatorname{sgn} x \cdot e^{-\frac{|x|}{k}}} \xrightarrow{*} T_{\operatorname{sgn} x}$

$$\operatorname{sgn} x \cdot e^{-\frac{|x|}{k}} \in L^1 \Rightarrow \mathcal{F}\left(T_{\operatorname{sgn} x \cdot e^{-\frac{|x|}{k}}}\right) = T_{\mathcal{F}\left(\operatorname{sgn} x \cdot e^{-\frac{|x|}{k}}\right)}$$

$$\mathcal{F}\left(\operatorname{sgn} x \cdot e^{-\frac{|x|}{k}}\right) = \int_{-\infty}^{\infty} \underbrace{\operatorname{sgn} x \, e^{-\frac{|x|}{k}}}_{\text{lichá}} \cdot \underbrace{e^{-2\pi i x \xi}}_{= \underbrace{\cos(2\pi x\xi)}_{\text{sudá}} - i \underbrace{\sin(2\pi x \xi)}_{\text{lichá}}} dx \qquad = -2i \int_0^{\infty} e^{-\frac{x}{k}} \sin(2\pi x \xi)\, dx$$

$$= 2i \operatorname{Im} \int_0^{\infty} e^{-\frac{x}{k}} e^{-2\pi x \xi} dx = 2i \operatorname{Im} \frac{1}{\frac{1}{k} + 2\pi i \xi} = \frac{-4\pi \xi i}{\frac{1}{k^2} + 4\pi^2 \xi^2}$$

Pro $k \to \infty$ je $\dfrac{-4\pi\xi i}{\frac{1}{k^2}+4\pi^2\xi^2} \longrightarrow -\dfrac{i}{\pi}\cdot\dfrac{1}{\xi}$, tj. $T_{\frac{-4\pi\xi i}{\frac{1}{k^2}+4\pi^2\xi^2}} \xrightarrow{*} -\frac{i}{\pi} T_{\text{p.v.}\frac{1}{\xi}}$ v S'

Raději si to ověříme: $\displaystyle \text{p.v.} \int_{-\infty}^{\infty} \left(\frac{-4\pi\xi i}{\frac{1}{k^2}+4\pi^2\xi^2} + \frac{i}{\pi}\frac{1}{\xi}\right)\varphi(\xi)\,d\xi = \frac{i}{\pi}\, \text{p.v.} \int_{-\infty}^{\infty} \left(\frac{1}{\xi} - \frac{4\pi^2\xi}{\frac{1}{k^2}+4\pi^2\xi^2}\right)\varphi(\xi)\,d\xi =$

$$= \frac{i}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{\frac{1}{k^2}\varphi(\xi)}{\xi\left(\frac{1}{k^2}+4\pi^2\xi^2\right)} d\xi = \underbrace{\int_{\mathbb{R}\setminus[-1,1]} \frac{\frac{1}{k^2}\varphi(\xi)}{\xi\left(\frac{1}{k^2}+4\pi^2\xi^2\right)} d\xi}_{\underbrace{\frac{1}{k^2}}_{\to 0} \underbrace{\int \dots}_{\text{omezené}} \Rightarrow \to 0} + \int_{-1}^{1} \underbrace{\frac{\frac{1}{k^2}}{\frac{1}{k^2}+4\pi^2\xi^2}}_{\le 1} \underbrace{\frac{\varphi(\xi)-\varphi(0)}{\xi}}_{\text{omezená fce}} d\xi + \underbrace{\text{p.v.}\int_{-1}^{1} \frac{\frac{1}{k^2}\varphi(0)}{\xi\left(\frac{1}{k^2}+4\pi^2\xi^2\right)} d\xi}_{\text{integrál z liché fce } = 0 \,!}$$

$\dfrac{\varphi(\xi)}{4\pi^2\xi^3}$ ... majoranta

$\left.\begin{array}{l} \Rightarrow \text{Lx. majoranta a} \\ \lim \dfrac{1/k^2}{1/k^2+4\pi^2\xi^2} = 0 \end{array}\right\} \Rightarrow \to 0$

Máme tak $\mathcal{F}(T_{\operatorname{sign} x}) = -\frac{i}{\pi} T_{\text{p.v.}\frac{1}{\xi}} = \frac{1}{i\pi} T_{\text{p.v.}1/\xi}$

Zpětnou F.T. bychom spočítali analogicky, bude se lišit jen ve znaménku: $\mathcal{F}^{-1}(T_{\operatorname{sign} x}) = \frac{i}{\pi} T_{\text{p.v.}1/\xi}$

a odtud $\mathcal{F}(\mathcal{F}^{-1}(T_{\operatorname{sign} x})) = \frac{i}{\pi}\mathcal{F}(T_{\text{p.v.}1/\xi}) = T_{\operatorname{sign} x} \Rightarrow \mathcal{F}(T_{\text{p.v.}1/\xi}) = -i\pi\, T_{\operatorname{sign} x}$

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21) Speciální případ 18) pro $m=2$.

Lze též: $H_{x^{-2}} = -D\,T_{\text{p.v.}\frac{1}{x}} \Rightarrow \mathcal{F}(H_{x^{-2}}) = -(2\pi i\xi)\,\mathcal{F}(T_{\text{p.v.}\frac{1}{x}}) = -2\pi i \xi\cdot(-i\pi)\,T_{\operatorname{sign}\xi} = -2\pi^2 \xi\, T_{\operatorname{sign}\xi}$

$= -2\pi^2 T_{|\xi|}$

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22) $H_{(x+i0)^\lambda} = H_{x_+^\lambda} + e^{i\lambda\pi} H_{x_-^\lambda}$ a použijeme př. 12) a 15)

$$\Rightarrow \mathcal{F}\left(H_{(x+i0)^\lambda}\right) = \Gamma(\lambda+1)\cdot(2\pi)^{-\lambda-1}\cdot\left[e^{-i(\lambda+1)\pi/2} H_{(\xi-i0)^{-\lambda-1}} + e^{i(3\lambda+1)\pi/2} H_{(\xi+i0)^{-\lambda-1}}\right] =$$

$$= \Gamma(\lambda+1)\cdot(2\pi)^{-\lambda-1}\cdot\left[e^{-i(\lambda+1)\pi/2}\cdot\left(H_{\xi_+^{-\lambda-1}} + e^{i(\lambda+1)\pi} H_{\xi_-^{-\lambda-1}}\right) + e^{i(3\lambda+1)\pi/2}\left(H_{\xi_+^{-\lambda-1}} + e^{-i(\lambda+1)\pi} H_{\xi_-^{-\lambda-1}}\right)\right] =$$

$$= \Gamma(\lambda+1)(2\pi)^{-\lambda-1}\cdot\left[H_{\xi_+^{-\lambda-1}}\cdot\left(e^{i\lambda\pi/2}\cdot\left(ie^{i\lambda\pi} - ie^{i\lambda\pi}\right)\right) + H_{\xi_-^{-\lambda-1}}\cdot\left(e^{i\lambda\pi/2}\cdot(i-i)\right)\right]$$

$$= \Gamma(\lambda+1)(2\pi)^{-\lambda-1}\cdot(-2)\sin\lambda\pi\cdot e^{i\lambda\pi/2}\cdot H_{\xi_+^{-\lambda-1}}$$

Eulerova reflexní formule: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

$$\Rightarrow \Gamma(1+\lambda)\Gamma(-\lambda) = \frac{\pi}{\sin(\pi+\pi\lambda)} = \frac{-\pi}{\sin\pi\lambda} \to \sin\pi\lambda = -\frac{\pi}{\Gamma(1+\lambda)\Gamma(-\lambda)}$$

$$\Rightarrow \mathcal{F}(H_{(x+i0)^\lambda}) = H_{\xi_+^{-\lambda-1}} \cdot e^{i\lambda\pi/2} \cdot \frac{2\pi}{\Gamma(-\lambda)(2\pi)^{\lambda+1}} = H_{\xi_+^{-\lambda-1}} \cdot e^{i\lambda\pi/2} \cdot (2\pi)^{-\lambda} \cdot \frac{1}{\Gamma(-\lambda)}$$

23) Analogicky jako 22) $H_{(x-i0)^\lambda} = H_{x_+^\lambda} + e^{-i\lambda\pi} H_{x_-^\lambda}$

Místo $e^{i(3\lambda+1)\pi/2}$ tak bude $e^{i(-\lambda+1)\pi/2}$.

Distribuci $H_{\xi_+^{-\lambda-1}}$ tak bude násobit $e^{-i(\lambda+1)\pi/2} + e^{i(-\lambda+1)\pi/2} = e^{-i\lambda\pi/2}\cdot\underbrace{\left[e^{-i\pi/2}+e^{i\pi/2}\right]}_{=-i+i=0} = 0$

Distribuci $H_{\xi_-^{-\lambda-1}}$ bude násobit $e^{i(\lambda+1)\pi/2} + e^{-i(3\lambda+1)\pi/2} = ie^{i\lambda\pi/2} - ie^{-3i\lambda\pi/2} = ie^{-i\lambda\pi/2}\left(e^{i\lambda\pi} - e^{-i\lambda\pi}\right)$

$= -2\sin\lambda\pi \cdot e^{-i\lambda\pi/2}$.

Konec opět Eulerovou formulí

24) Vyžaduje aparát o transformacích radiálně symetrických fcí. To je detailně popsáno ve skriptech a příklad je přímo vyřešen v příslušné kapitole.

Dodatek: Konvoluční rovnice!

Prostřednictvím Fourierovy transformace se řeší příklady typu: Najděte funkci / distribuci f tak, že pro dané jádro G a danou funkci / distribuci g platí

f * G = g

Řešení: Dvě možnosti.

1) Aplikace FT: F(f)F(G) = F(g)
F(f) = F(g)/F(G)
f = F^{-1} (F(g)/F(G))

2) Přes fundamentální řešení. Hledáme E t.ž.
E * G = delta (Dirac) (tj. F(E)F(G) = 1)
pak f = E * g

Možné příklady: f * exp(-\alpha |x|) = \delta nebo f * exp(-\alpha |x|) = exp(-\beta |x|)
f * (v.p. 1/x) = \delta nebo f * (v.p. 1/x) = sin(\alpha x)/x