Sada příkladů 10/11

Distribuce

1. Zjednodušte zápis distribuce $T \in \mathcal{D}'(\mathbb{R})$:
   a) $x^k D^n \delta_0, \quad k, n \in \mathbb{N}$ b) $\mathrm{e}^{\mathrm{i}x\omega} D^n \delta_0, \quad \omega \in \mathbb{R}, n \in \mathbb{N}$

2. Zjednodušte zápis distribuce $T \in \mathcal{D}'(\mathbb{R}^N)$:
   a) $|x|^2 \Delta \delta_0$ b) $\mathrm{e}^{\mathrm{i}(x,\omega)} \Delta^k \delta_0, \quad \omega \in \mathbb{R}^N, k \in \mathbb{N}$
   c) $\mathrm{e}^{-a|x|^2} \Delta \delta_0, \quad a > 0$

3. Určete distribuce $\Delta T_u \in \mathcal{D}'(\mathbb{R}^N)$:
   a) $u(x) = |x|^\lambda, \quad \lambda \geq 2 - N, N \geq 2$ b) $u(x) = \ln |x|$

4. Dokažte: Nechť f je hladká funkce na $\mathbb{R} \setminus \{0\}$, $A_k = f_+^{(k)}(0) - f_-^{(k)}(0)$, $k = 0, 1, \ldots, n-1$. Potom
$$D^n T_f = T_{f^{(n)}} + \sum_{k=0}^{n-1} A_k D^{n-1-k} \delta_0.$$

5. Ukažte, že posloupnosti
   a) $f_n(x) = \frac{1}{\pi} \frac{n}{n^2 x^2 + 1}$ b) $g_n(x) = \frac{n}{2\sqrt{\pi}} \mathrm{e}^{-\frac{n^2 x^2}{4}}$ c) $h_n(x) = \frac{1}{\pi} \frac{\sin(nx)}{x}$
   konvergují v $\mathcal{D}'(\mathbb{R})$ k δ_0 distribuci.

6. Ukažte, že
$$T_{\frac{1}{x - \mathrm{i}0}} = T_{\mathrm{p.v.}\frac{1}{x}} + \mathrm{i}\pi\delta_0.$$

7. Nalezněte rozvoj do Fourierových řad pro periodické distribuce:
   a) $T_{\mathrm{p.v.cotg}(\pi x)}$ b) $T_{\mathrm{p.v.tg}(\pi x)}$ c) $T_{\mathrm{p.v.}\frac{1}{\sin(\pi x)}}$

8. Dokažte, že:
   a) $\delta_0 \circ (\mathbb{A}x) = \dfrac{1}{|\det \mathbb{A}|} \delta_0$
   b) $\delta_0 \circ (x + \mathbf{b}) = \delta_{\mathbf{b}}$
   c) $\delta_0 \circ (ax) = \dfrac{1}{|a|^N} \delta_0.$

9. Ukažte, že metoda zavedení distribucí $H_{x_+^\lambda}$ pomocí Taylorova rozvoje testovacích funkcí dává totéž co holomorfní rozšiřování.

10. Ukažte, že limity

$$\lim_{\lambda\to -2m} H_{|x|^\lambda} \qquad \lim_{\lambda\to -2m+1} H_{|x|^\lambda \operatorname{sign} x}$$

existují v $\mathcal{D}'(\mathbb{R})$ a tudíž definují distribuce $H_{x^{-2m}}$ resp. $H_{x^{-2m+1}}$.

11. Dokažte pro $k \in \mathbb{N}$

$$\begin{aligned} H_{(x\pm \mathrm{i}0)^{-k}} &= H_{x^{-k}} \mp \frac{\mathrm{i}\pi(-1)^{k-1}}{(k-1)!} D^{k-1}\delta_0 \\ H_{x^{-k}} &= \frac{1}{2}\Big(H_{(x+\mathrm{i}0)^{-k}} + H_{(x-\mathrm{i}0)^{-k}}\Big) \\ H_{(x+\mathrm{i}0)^{-k}} - H_{(x-\mathrm{i}0)^{-k}} &= -2\pi\mathrm{i}\frac{(-1)^{k-1}}{(k-1)!} D^{k-1}\delta_0. \end{aligned}$$

Fourierovy řady vs. distribuce

$f \in L^1((0,1))$, její F. koeficienty $c_n := \int_0^1 f(x)\, e^{2\pi i n x}\, dx$, $n \in \mathbb{Z}$

a řada je $f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$. Pro $f \in L^1$ ale neplatí rovnost mezi f a řadou.

Můžeme ale integrovat ! $F(x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{c_n}{2\pi i n} e^{2\pi i n x} + c_0 x + d_0$.

[Můžeme integrovat tak dlouho, dokud řada vpravo nebude konvergovat stejnoměrně !, což je v případech kdy c_n roste jako polynom v n.]

Nyní $\langle T_F, \varphi \rangle = \int_{\mathbb{R}} F\varphi = \lim_{m\to\infty} \int_{\mathbb{R}} \Big(\sum_{\substack{n=-m \\ n\neq 0}}^{m} \frac{c_n}{2\pi i n} e^{2\pi i n x} + c_0 x + d_0 \Big) \varphi\, dx$ [prohodili jsme lim a $\int$, řada totiž konverguje stejnoměrně]

a
$$\left.\begin{aligned} \langle DT_F, \varphi \rangle &= -\langle T_F, \varphi' \rangle = -\int_{\mathbb{R}} F\varphi' = \int_{\mathbb{R}} F'\varphi = \int_{\mathbb{R}} f\varphi = \langle T_f, \varphi \rangle \\ &\;\; \| \\ -\lim_{m\to\infty} \int_{\mathbb{R}} \Big(\sum_{\substack{n=-m \\ n\neq 0}}^{m} & \frac{c_n}{2\pi i n} e^{2\pi i n x} + c_0 x + d_0 \Big) \varphi'\, dx = \lim_{m\to\infty} \int_{\mathbb{R}} \Big(\sum_{n=-m}^{m} c_n e^{2\pi i n x} \Big) \varphi\, dx \end{aligned}\right\} \forall \varphi \in \mathcal{D}(\mathbb{R})$$

a odtud $T_f = \lim_{m\to\infty} T_{\sum_{-m}^{m} c_n e^{2\pi i n x}}$

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7 a) Začneme řadou $\sum_{n=0}^{\infty} e^{2\pi i n x}$, tedy $c_n = 1$ pro $n \geq 0$, $c_n = 0$ pro $n < 0$

Pak $s_n = \sum_{k=0}^{n} e^{2\pi i k x} = \frac{1 - e^{2\pi i x (n+1)}}{1 - e^{2\pi i x}}$ Zřejmě $\int_{-1/2}^{1/2} s_n(x)\, dx = 1$ $\forall n \in \mathbb{N}_0$, jediný nenulový integrál je totiž z členu $e^{2\pi i x \cdot 0}$

$$\langle T_{s_n(x)}, \varphi \rangle = \Big\langle \frac{1 - e^{2\pi i x (n+1)}}{1 - e^{2\pi i x}}, \varphi \Big\rangle = \int_{-1/2}^{1/2} \frac{1 - e^{2\pi i x (n+1)}}{1 - e^{2\pi i x}} \cdot (\varphi(x) - \varphi(0))\, dx + \underbrace{\varphi(0) \int_{-1/2}^{1/2} s_n(x)\, dx}_{= \varphi(0)}$$

$\varphi \in \mathcal{D}((-1/2, 1/2))$

$$\int_{-1/2}^{1/2} \frac{1}{1 - e^{2\pi i x}} \cdot (\varphi(x) - \varphi(0))\, dx - \underbrace{\int_{-1/2}^{1/2} \frac{e^{2\pi i x (n+1)}}{1 - e^{2\pi i x}} \cdot (\varphi(x) - \varphi(0))\, dx}$$

Toto je $\tilde{c}_{n+1}$ od funkce $\frac{\varphi(x) - \varphi(0)}{1 - e^{2\pi i x}}$, ta má v 0 konečnou limitu $\Rightarrow$ lze dodefinovat a je $L^2((-1/2, 1/2))$

Riemann-Lebesgue: $\tilde{c}_{n+1} \to 0$

$$\frac{1}{1 - e^{2\pi i x}} = \frac{1}{1 - \cos 2\pi x - i \sin 2\pi x} = \frac{1 - \cos 2\pi x + i(2 \sin \pi x \cos \pi x)}{1 - 2\cos 2\pi x + \cos^2 2\pi x + \sin^2 2\pi x} =$$
$$= \frac{1}{2} \cdot \Big(1 + i \frac{2 \sin \pi x \cos \pi x}{1 - \cos 2\pi x} \Big) = \frac{1}{2} (1 + i \operatorname{cotg} \pi x)$$

a odtud $\int_{-1/2}^{1/2} \frac{\varphi(x) - \varphi(0)}{1 - e^{2\pi i x}}\, dx = \lim_{\varepsilon\to 0_+} \frac{1}{2} \int_{\varepsilon < |x| < 1/2} (\varphi(x) - \varphi(0))(1 + i \operatorname{cotg} \pi x)$

$$= \frac{1}{2} \int_{-1/2}^{1/2} \varphi(x)\, dx - \frac{1}{2}\varphi(0) + \frac{i}{2} \lim_{\varepsilon\to 0_+} \int_{\varepsilon < |x| < 1/2} \operatorname{cotg} \pi x\, \varphi(x)\, dx$$ (poslední člen vypadne, $\operatorname{cotg} \pi x$ je lichá fce)

$$\Rightarrow \langle T_{s_n(x)}, \varphi \rangle \xrightarrow{n\to\infty} \frac{1}{2}\varphi(0) + \frac{1}{2} \langle T_1, \varphi \rangle + \frac{i}{2} \langle T_{\text{p.v.} \operatorname{cotg} \pi x}, \varphi \rangle$$

Reálná část: $\sum_{n=0}^{\infty} T_{\cos 2\pi n x} = \frac{1}{2}\delta_0 + \frac{1}{2} T_1$ Imaginární část: $\sum_{n=0}^{\infty} T_{\sin 2\pi n x} = \frac{1}{2} T_{\text{p.v.} \operatorname{cotg} \pi x}$

OBOJÍ VE SMYSLU ROVNOSTI DISTRIBUCÍ V $\mathcal{D}'((-1/2, 1/2))$

b) Použijeme vztah $\operatorname{cotg}(A-\frac{\pi}{2}) = -\operatorname{tg} A$, čímž vše posuneme z intervalu $(-\frac{1}{2},\frac{1}{2})$ na $(0,1)$.

Je-li $\varphi \in \mathcal{D}((-\frac{1}{2},\frac{1}{2}))$, pak $\tilde{\varphi}(y) := \varphi(y-\frac{1}{2}) \in \mathcal{D}((0,1))$

Máme $\forall \varphi \in \mathcal{D}((-\frac{1}{2},\frac{1}{2}))$:

$$\langle T_{\text{p.v.}\operatorname{cotg}\pi x}, \varphi\rangle = \text{p.v.}\int_{-1/2}^{1/2} \operatorname{cotg}\pi x\, \varphi(x)\,dx \overset{y=x+1/2}{=} \text{p.v.}\int_0^1 \operatorname{cotg}(\pi y - \tfrac{\pi}{2})\,\varphi(y-\tfrac{1}{2})\,dy = -\text{p.v.}\int_0^1 \operatorname{tg}(\pi y)\,\tilde{\varphi}(y)\,dy = -\langle T_{\text{p.v.}\operatorname{tg}\pi x}, \tilde{\varphi}\rangle$$

$$\langle T_{\text{p.v.}\operatorname{cotg}\pi x}, \varphi\rangle = \lim_{m\to\infty}\sum_0^m \langle T_{\sin 2\pi n x}, \varphi\rangle = \lim_{m\to\infty}\sum_{n=0}^m \int_{-1/2}^{1/2} \sin 2\pi n x\, \varphi(x)\,dx = \lim_{m\to\infty}\sum_0^m \int_0^1 \sin(2\pi n y - \pi n)\,\varphi(y-\tfrac{1}{2})\,dy = \lim_{m\to\infty}\sum_0^m\int_0^1 (-1)^n \sin 2\pi n y\,\tilde{\varphi}(y)\,dy = \Big\langle \sum_0^\infty T_{(-1)^n \sin 2\pi n y}, \tilde{\varphi}\Big\rangle$$

a odtud tedy $T_{\text{p.v.}\operatorname{tg}\pi x} = \sum_0^\infty T_{(-1)^{n+1}\sin 2\pi n y}$ ve smyslu distribucí v $\mathcal{D}'((0,1))$.

c) Použijeme vztah $\frac{1}{\sin A} = \operatorname{cotg}\frac{A}{2} - \operatorname{cotg} A$ a tedy pro interval $(-1,1)$ dostaneme

$$\langle T_{\text{p.v.}\frac{1}{\sin\pi x}}, \varphi\rangle = \text{p.v.}\int_{-1}^1 \frac{1}{\sin\pi x}\varphi(x)\,dx = \text{p.v.}\int_{-1}^1 \operatorname{cotg}\frac{\pi x}{2}\varphi(x)\,dx - \text{p.v.}\int_{-1}^1 \operatorname{cotg}\pi x\,\varphi(x)\,dx$$

$$= \text{p.v.}\,2\int_{-1/2}^{1/2} \operatorname{cotg}\pi y\,\varphi(2y)\,dy - \text{p.v.}\int_{-1}^1 \operatorname{cotg}\pi x\,\varphi(x)\,dx$$

$$= \lim_{m\to\infty}\sum_{n=0}^m 4\int_{-1/2}^{1/2} \sin 2\pi n y\,\varphi(2y)\,dy - \lim_{m\to\infty}\sum_{n=0}^m 2\int_{-1}^1 \sin 2\pi n x\,\varphi(x)\,dx$$

$$= \lim_{m\to\infty}\sum_{n=0}^m 2\int_{-1}^1 \sin \pi n x\,\varphi(x)\,dx - \lim_{m\to\infty}\sum_0^m 2\int_{-1}^1 \sin 2\pi n x\,\varphi(x)\,dx$$

$$= \lim_{m\to\infty}\sum_{n=0}^m 2\int_{-1}^1 \sin(2n+1)\pi x\,\varphi(x)\,dx = 2\sum_0^\infty\langle T_{\sin(2n+1)\pi x}, \varphi\rangle$$

a tedy $T_{\text{p.v.}\frac{1}{\sin\pi x}} = 2\sum_0^\infty T_{\sin(2n+1)\pi x}$ v $\mathcal{D}'((-1,1))$

SKLÁDÁNÍ DISTRIBUCÍ S DIFEOMORFISMY

$\Omega \subset \mathbb{R}^n$ otevřená, $h: \Omega \to \mathbb{R}^n$ je C^∞ difeomorfismus zobrazující Ω na $\tilde{\Omega}$. Nechť $T \in \mathcal{D}'(\tilde{\Omega})$. Pak $T\circ h \in \mathcal{D}'(\Omega)$ je definováno jako $\langle T\circ h, \varphi\rangle = \langle T, \frac{1}{|J_h(h^{-1}(y))|}\cdot\varphi(h^{-1}(y))\rangle$ $\forall \varphi \in \mathcal{D}(\Omega)$

8a) $\langle \delta_0\circ(Ax), \varphi\rangle = \langle \delta_0, \frac{1}{|\det A|}\varphi(A^{-1}x)\rangle = \frac{1}{|\det A|}\langle\delta_0, \varphi(A^{-1}x)\rangle = \frac{1}{|\det A|}\varphi(0) = \langle \frac{1}{|\det A|}\delta_0, \varphi\rangle$ $\forall\varphi\in\mathcal{D}(\mathbb{R}^N)$

8b) $\langle \delta_0\circ(x+b), \varphi\rangle = \langle \delta_0, \varphi(x-b)\rangle = \varphi(b) = \langle\delta_b, \varphi\rangle$ $\forall\varphi\in\mathcal{D}(\mathbb{R}^N)$

8c) $\langle \delta_0\circ(ax), \varphi\rangle = \langle \delta_0, \frac{1}{|a|^N}\varphi(\frac{x}{a})\rangle = \frac{1}{|a|^N}\varphi(0) = \frac{1}{|a|^N}\langle\delta_0, \varphi\rangle$ $\forall\varphi\in\mathcal{D}(\mathbb{R}^N)$

Homogenní distribuce

$\lambda \in \mathbb{C}$ a zavádíme $x_+^\lambda := \begin{cases} x^\lambda & \text{pro } x>0 \\ 0 & \text{pro } x\le 0 \end{cases}$ a $x_-^\lambda := (-x)_+^\lambda = \begin{cases} 0 & \text{pro } x\ge 0 \\ |x|^\lambda & \text{pro } x<0. \end{cases}$

$\operatorname{Re}\lambda > -1 \Rightarrow$ existuje regulární distribuce $T_{x_+^\lambda}$, protože $x_+^\lambda \in L^1_{loc}(\mathbb{R})$

$\operatorname{Re}\lambda \le -1$??

<u>Definice</u> $k \in \mathbb{N}$. Pak na $G_k := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -k-1,\ -\lambda \notin \mathbb{N}\}$ definujeme parametrické systémy distribucí $\{H_{x_+^\lambda}\}_{\lambda\in G_k}$ a $\{H_{x_-^\lambda}\}_{\lambda\in G_k}$ pomocí předpisu

$$H_{x_+^\lambda} = \frac{D^k T_{x_+^{\lambda+k}}}{(\lambda+1)(\lambda+2)\dots(\lambda+k)} \quad \text{a} \quad \langle H_{x_-^\lambda},\varphi\rangle = \langle H_{x_+^\lambda},\varphi(-x)\rangle \quad \forall \varphi\in\mathcal{D}(\mathbb{R})$$

<u>Platí</u>:
- $x H_{x_+^\lambda} = H_{x_+^{\lambda+1}}$ a $-x H_{x_-^\lambda} = H_{x_-^{\lambda+1}}$
- $H_{x_+^0} = T_{x_+^0} = T_H$, kde H je Heavisidova fce
- $\forall \lambda \in \mathbb{C}\setminus\{-n;\ n\in\mathbb{N}_0\}$: $DH_{x_+^\lambda} = \lambda H_{x_+^{\lambda-1}}$ a $DH_{x_-^\lambda} = -\lambda H_{x_-^{\lambda-1}}$
- $\forall k\in\mathbb{N}$ mají systémy $\{H_{x_+^\lambda}\}$ a $\{H_{x_-^\lambda}\}$ v bodě $-k$ izolovanou singularitu a

$$\operatorname{Res}_{-k} H_{x_+^\lambda} = (-1)^{k-1}\frac{D^{k-1}\delta_0}{(k-1)!} \quad \text{a} \quad \operatorname{Res}_{-k} H_{x_-^\lambda} = \frac{D^{k-1}\delta_0}{(k-1)!}$$

(to znamená, že $\langle \operatorname{Res}_{-k\lambda} H_{x_+^\lambda},\varphi\rangle = \operatorname{Res}_{-k}\langle H_{x_+^\lambda},\varphi\rangle$ atd.)

$T_{x_+^\lambda}$ lze ale rozšířit i jinak:

$$\langle T_{x_+^\lambda},\varphi\rangle = \int_0^\infty x^\lambda\varphi(x)\,dx = \int_0^1 x^\lambda\varphi(x)\,dx + \int_1^\infty x^\lambda\varphi(x)\,dx = \underbrace{\int_0^1 x^\lambda(\varphi(x)-\varphi(0))\,dx + \frac{\varphi(0)}{\lambda+1} + \int_1^\infty x^\lambda\varphi(x)\,dx}_{\text{toto je již dobře definováno pro } \operatorname{Re}\lambda>-2,\ \lambda\ne-1}$$

$$= \dots = \int_0^1 x^\lambda\cdot\left(\varphi(x)-\varphi(0)-\varphi'(0)x-\dots-\frac{\varphi^{(m-1)}(0)}{(m-1)!}x^{m-1}\right)dx + \sum_{k=1}^{m}\frac{\varphi^{(k-1)}(0)}{(k-1)!(\lambda+k)} + \int_1^\infty x^\lambda\varphi(x)\,dx$$

a toto funguje pro $\operatorname{Re}\lambda > -m-1$, $\lambda\ne -k$, $k\in\mathbb{N}$.

9) Máme ukázat, že obě metody dají totéž. Nejprve $k=1$, tj. $\operatorname{Re}\lambda>-2$, $\lambda\ne-1$

$$\int_0^1 x^\lambda(\varphi(x)-\varphi(0))\,dx + \frac{\varphi(0)}{\lambda+1} + \int_1^\infty x^\lambda\varphi(x)\,dx = (\text{per partes}) = -\int_0^1\frac{x^{\lambda+1}}{\lambda+1}\varphi'(x)\,dx + \left[\frac{x^{\lambda+1}}{\lambda+1}(\varphi(x)-\varphi(0))\right]_0^1 + \frac{\varphi(0)}{\lambda+1}$$

$$-\int_1^\infty\frac{x^{\lambda+1}}{\lambda+1}\varphi'(x)\,dx + \left[\frac{x^{\lambda+1}}{\lambda+1}\varphi(x)\right]_1^\infty$$

$$= -\int_0^\infty\frac{x^{\lambda+1}}{\lambda+1}\varphi'(x)\,dx = -\frac{1}{\lambda+1}\langle x_+^{\lambda+1},\varphi'\rangle = \frac{1}{\lambda+1}\langle D x_+^{\lambda+1},\varphi\rangle = \langle H_{x_+^\lambda},\varphi\rangle \quad \text{pro } k=1$$

Dále indukcí. Nechť to platí pro $k-1$, ukážeme, že to platí pro k. Nechť $\operatorname{Re}\lambda > -1-k$

$$\int_0^1 x^\lambda \Big(\varphi(x) - \sum_{j=1}^{k} \frac{\varphi^{(j-1)}(0)\, x^{j-1}}{(j-1)!}\Big)dx + \sum_{j=1}^{k} \frac{\varphi^{(j-1)}(0)}{(j-1)!(\lambda+j)} + \int_1^\infty x^\lambda \varphi(x)\, dx$$

$$= \int_0^1 x^\lambda \Big(\varphi(x) - \sum_{j=1}^{k-1} \frac{\varphi^{(j-1)}(0)\, x^{j-1}}{(j-1)!}\Big)dx + \sum_{j=1}^{k-1} \frac{\varphi^{(j-1)}(0)}{(j-1)!(\lambda+j)} + \int_1^\infty x^\lambda \varphi(x)\, dx - \int_0^1 x^\lambda \frac{\varphi^{(k-1)}(0)\, x^{k-1}}{(k-1)!}dx + \frac{\varphi^{(k-1)}(0)}{(k-1)!(\lambda+k)}$$

$$= \text{dle I.P.} = \frac{(-1)^{k-1}}{(\lambda+1)(\lambda+2)\dots(\lambda+k-1)} \langle x_+^{\lambda+k-1}, \varphi^{(k-1)} \rangle \underbrace{- \frac{\varphi^{(k-1)}(0)}{(k-1)!}\Big[\frac{x^{\lambda+k}}{\lambda+k}\Big]_0^1 + \frac{\varphi^{(k-1)}(0)}{(k-1)!(\lambda+k)}}_{=0}$$

$$\overset{\text{slabá derivace}}{=} \frac{(-1)^k}{(\lambda+1)\dots(\lambda+k)} \langle x_+^{\lambda+k}, \varphi^{(k)} \rangle = \langle H_{x_+^\lambda}, \varphi \rangle$$

Mohli jsme, protože $\operatorname{Re}\lambda > -k-1$!

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10) Pozorování: k celé, $k = -1, -2, \dots \Rightarrow$ lze definovat $\langle T_{v.p.\frac{1}{x}}, \varphi \rangle := v.p. \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx$

a dále $\langle H_{-\frac{1}{x^2}}, \varphi \rangle = \langle D T_{v.p.\frac{1}{x}}, \varphi \rangle$ atd $\langle H_{\frac{1}{x^k}}, \varphi \rangle = \Big\langle \frac{D H_{\frac{1}{x^{k-1}}}}{1-k}, \varphi \Big\rangle$, což odpovídá definici $H_{x_+^\lambda}$

Nyní: $|x|^\lambda = x_+^\lambda + x_-^\lambda$ a $|x|^\lambda \operatorname{sgn} x = x_+^\lambda - x_-^\lambda$, přičemž $\langle H_{x_-^\lambda}, \varphi \rangle = \langle H_{x_+^\lambda}, \varphi(-x) \rangle$

Pro $\operatorname{Re}\lambda > -2m-1$, $\lambda \neq -1, -2, \dots, -2m$ je $\langle H_{x_+^\lambda}, \varphi \rangle = (-1)^{2m} \dfrac{\langle H_{x_+^{\lambda+2m}}, \varphi^{(2m)} \rangle}{(\lambda+1)\dots(\lambda+2m)}$

derivace fce $-x$

a $\langle H_{x_-^\lambda}, \varphi \rangle = (-1)^{2m} \dfrac{\langle H_{x_+^{\lambda+2m}}, \varphi^{(2m)}(-x) \rangle}{(\lambda+1)\dots(\lambda+2m)} \cdot (-1)^{2m}$

$$\Rightarrow \langle H_{x_+^\lambda + x_-^\lambda}, \varphi \rangle = \int_0^\infty \frac{x^{\lambda+2m} \cdot \big(\varphi^{(2m)}(x) + \varphi^{(2m)}(-x)\big)}{(\lambda+1)\dots(\lambda+2m)} dx = \lim_{\varepsilon \to 0_+} \int_\varepsilon^\infty \frac{x^{\lambda+2m} \cdot \big(\varphi^{(2m)}(x) + \varphi^{(2m)}(-x)\big)}{(\lambda+1)\dots(\lambda+2m)} dx \overset{\text{per partes}}{=}$$

$$= \lim_{\varepsilon \to 0_+} \Bigg(\Big[\frac{x^{\lambda+2m} \varphi^{(2m-1)}(x)}{(\lambda+1)\dots(\lambda+2m)}\Big]_\varepsilon^\infty - \Big[\frac{x^{\lambda+2m} \varphi^{(2m-1)}(-x)}{(\lambda+1)\dots(\lambda+2m)}\Big]_\varepsilon^\infty - \int_\varepsilon^\infty \frac{x^{\lambda+2m-1}\big(\varphi^{(2m-1)}(x) - \varphi^{(2m-1)}(-x)\big)}{(\lambda+1)\dots(\lambda+2m-1)} dx \Bigg) = (*)$$

Hranaté závorky v ∞: $= 0$, protože $\varphi \in \mathcal{D}(\mathbb{R})$

Hranaté závorky v ε: $\lim_{\varepsilon \to 0_+} \varepsilon^{\lambda+2m} \cdot \big(\varphi^{(2m-1)}(\varepsilon) - \varphi^{(2m-1)}(-\varepsilon)\big) = \lim_{\varepsilon \to 0_+} 2 \cdot \varepsilon^{\lambda+2m+1} \cdot \underbrace{\dfrac{\varphi^{(2m-1)}(\varepsilon) - \varphi^{(2m-1)}(-\varepsilon)}{2\varepsilon}}_{\to \varphi^{(2m)}(0)} = 0$

$\varepsilon^{\lambda+2m+1} \to 0$ protože $\operatorname{Re}\lambda > -2m-1$

$$(*) = -\lim_{\varepsilon \to 0_+} \int_\varepsilon^\infty \frac{x^{\lambda+2m-1}\big(\varphi^{(2m-1)}(x) - \varphi^{(2m-1)}(-x)\big)}{(\lambda+1)\cdot\ldots\cdot(\lambda+2m-1)} dx = -\lim_{\varepsilon \to 0_+} \int_\varepsilon^\infty \frac{x^{\lambda+2m}}{(\lambda+1)\cdot\ldots\cdot(\lambda+2m-1)} \cdot \frac{\varphi^{(2m-1)}(x) - \varphi^{(2m-1)}(-x)}{x} dx$$

$$= -\int_0^\infty \frac{x^{\lambda+2m}}{(\lambda+1)\cdot\ldots\cdot(\lambda+2m-1)} \cdot \underbrace{\frac{\varphi^{(2m-1)}(x) - \varphi^{(2m-1)}(-x)}{x}}_{\text{má konečnou limitu v nule, integrál tak u 0 konverguje pro } \operatorname{Re}\lambda > -2m-1} dx.$$

Pro $\lambda \to -2m$ je $x^{\lambda+2m} \to 1$

Na okolí $\lambda = -2m$ je majoranta $\Rightarrow$ prohodíme $\lim_{\lambda \to -2m}$ a $\int$ a dostaneme

$\langle H_{x^{-2m}}, \varphi \rangle = -\displaystyle\int_0^\infty \frac{1}{(-2m+1)\cdot\ldots\cdot(-1)} \cdot \frac{\varphi^{(2m-1)}(x) - \varphi^{(2m-1)}(-x)}{x} dx$ a pro $m=1$: $\langle H_{x^{-2}}, \varphi \rangle = \displaystyle\int_{-\infty}^{\infty} \frac{\varphi'(x)}{x} dx = \langle T_{v.p.\frac{1}{x}}, \varphi' \rangle$

Analogicky pro $H_{x_+^\lambda - x_-^\lambda}$: Pro $\operatorname{Re}\lambda > -2m$ je $\langle H_{x_+^\lambda}, \varphi\rangle = (-1)^{-2m+1} \dfrac{\langle H_{x_+^{\lambda+2m-1}}, \varphi^{(2m-1)}\rangle}{(\lambda+1)\dots(\lambda+2m-1)}$

a $\langle H_{x_-^\lambda}, \varphi\rangle = (-1)^{-2m+1} \dfrac{\langle H_{x_+^{\lambda+2m-1}}, \varphi^{(2m-1)}(-x)\rangle}{(\lambda+1)\dots(\lambda+2m-1)} \cdot (-1)^{2m-1}$

$\Rightarrow \langle H_{x_+^\lambda - x_-^\lambda}, \varphi\rangle = -\int_0^\infty \dfrac{x^{\lambda+2m-1} \cdot (\varphi^{(2m-1)}(x) + \varphi^{(2m-1)}(-x))}{(\lambda+1)\cdot\ldots\cdot(\lambda+2m-1)}$. Jsme ve stejném bodě jako dříve, dál postupujeme stejně až k výsledku $\ldots = \int_0^\infty \dfrac{x^{\lambda+2m-1}}{(\lambda+1)\cdot\ldots\cdot(\lambda+2m-2)} \cdot \dfrac{\varphi^{(2m-2)}(x) - \varphi^{(2m-2)}(-x)}{x}\, dx$.

Opět díky Lebesgueově větě toto má limitu $\lambda \to -2m+1$ a dostaneme

$\langle H_{x^{-2m+1}}, \varphi\rangle = \int_0^\infty \dfrac{1}{(-2m+2)\cdot\ldots\cdot(-1)} \cdot \dfrac{\varphi^{(2m-2)}(x) - \varphi^{(2m-2)}(-x)}{x}\, dx$. (Nebo jako $\lim\limits_{\varepsilon\to 0_+} \int_\varepsilon^\infty \ldots$)

Speciálně tímto postupem pro $m=1$ skončíme s $\langle H_{x^{-1}}, \varphi\rangle = \lim\limits_{\varepsilon\to 0_+} \int_\varepsilon^\infty \dfrac{\varphi(x) - \varphi(-x)}{x}\, dx = \text{p.v.} \int_{-\infty}^{\infty} \dfrac{\varphi(x)}{x}\, dx$

$= \langle T_{\text{p.v.}\frac{1}{x}}, \varphi\rangle$

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11) $(x+i0)^{-k} = \lim\limits_{y\to 0_+} (x+iy)^{-k} = x^{-k}$ pro $x > 0$

a $= |x|^{-k} \cdot e^{-\pi i k}$ pro $x < 0$

Proto definujeme $H_{(x\pm i0)^\lambda} := H_{x_+^\lambda} + e^{\pm \pi i \lambda} H_{x_-^\lambda}$

$x+i0$: $\lambda \to -k$, k liché: $\lim\limits_{\lambda\to -k} (x_+^\lambda + e^{i\lambda\pi} x_-^\lambda) = \lim\limits_{\lambda\to -k} \Big(\underbrace{x_+^\lambda - x_-^\lambda}_{\to\, x^{-k} \text{ pro } k \text{ liché}} + \underbrace{\dfrac{e^{i\lambda\pi}+1}{\lambda+k}}_{\text{l'Hosp.} \to \frac{e^{ik\pi}\cdot i\pi}{1}} \overbrace{x_-^\lambda (\lambda+k)}^{\text{definice Res.}}\Big)$

$= x^{-k} - i\pi \cdot \operatorname{Res}_{-k} x_-^\lambda = x^{-k} - i\pi \dfrac{D^{k-1}\delta_0}{(k-1)!}$

$x+i0$: $\lambda \to -k$, k sudé: $\lim\limits_{\lambda\to -k} (x_+^\lambda + e^{i\lambda\pi} x_-^\lambda) = \lim\limits_{\lambda\to -k} \Big(\underbrace{x_+^\lambda + x_-^\lambda}_{\to\, x^{-k} \text{ pro } k \text{ sudé}} + \underbrace{\dfrac{e^{i\lambda\pi}-1}{\lambda+k}}_{\to \text{l'Hosp.: } \frac{i\pi e^{ik\pi}}{1}} \overbrace{x_-^\lambda (\lambda+k)}^{\text{Res}}\Big)$

$= x^{-k} + i\pi \operatorname{Res}_{-k} x_-^\lambda = x^{-k} + i\pi \dfrac{D^{k-1}\delta_0}{(k-1)!}$

$x-i0$: $\lambda \to -k$, k liché: $\lim\limits_{\lambda\to -k} (x_+^\lambda + e^{-i\lambda\pi} x_-^\lambda) = \lim\limits_{\lambda\to -k} \Big(x_+^\lambda - x_-^\lambda + \underbrace{\dfrac{e^{-i\lambda\pi}+1}{\lambda+k}}_{\to \frac{-i\pi e^{-ik\pi}}{1}} (\lambda+k) x_-^\lambda\Big)$

$= x^{-k} + i\pi \dfrac{D^{k-1}\delta_0}{(k-1)!}$

$x-i0$: $\lambda \to -k$, k sudé: $\lim\limits_{\lambda\to -k} (x_+^\lambda + e^{-i\lambda\pi} x_-^\lambda) = \lim\limits_{\lambda\to -k} \Big(x_+^\lambda + x_-^\lambda + \dfrac{e^{-i\lambda\pi}-1}{\lambda+k} (\lambda+k) x_-^\lambda\Big) = x^{-k} - i\pi \dfrac{D^{k-1}\delta_0}{(k-1)!}$

Tím jsou vyřešeny všechny případy a je dokázán první vztah. Zbylé jsou součtem, resp. rozdílem těch prvních.